Then express the system of constraint equations in the following matrix form :

$$\begin{array}{cccccc} \mathbf{e}_1 \; \beta_0^{(1)} & \mathbf{a}_1^{(1)} & \mathbf{a}_2^{(1)} & \mathbf{a}_3^{(1)} \; \beta_1^{(1)} & \mathbf{a}_4^{(1)} \; \beta_2^{(1)} & \mathbf{a}_5^{(1)} \; \beta_3^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 5 \\ 6 \end{bmatrix}$$

Now form the revised simplex table for the first iteration.

Table 6.7

Variables in the basis	$\beta_0^{(1)}$ $\mathbf{e}_1$	$\beta_1^{(1)}$ $(\mathbf{a}_3^{(1)})$	$\beta_2^{(1)}$ $(\mathbf{a}_4^{(1)})$	$\beta_3^{(1)}$ $(\mathbf{a}_5^{(1)})$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)}$ $(k=2)$	Min. $(\mathbf{X}_B/\mathbf{X}_2)$
z	1	0	0	0	0	−2	
$x_3 = x_{B1}$	0	1	0	0	3	1	3/1
$x_4 = x_{B2}$	0	0	1	0	5	$\boxed{2}$	5/2 ←
$x_5 = x_{B3}$	0	0	0	1	6	1	6/1
	$\underbrace{\qquad\qquad}_{\mathbf{B}_1^{-1}}$					↑	

Additional Table

$\mathbf{a}_1^{(1)}$	$\mathbf{a}_2^{(1)}$
−1	−2
1	1
1	2
3	1

First Iteration

Step 1. ***Compute Δ_j for $\mathbf{a}_1^{(1)}$ and $\mathbf{a}_2^{(1)}$, i.e. (Δ_1, Δ_2).***

$$\{\Delta_1, \Delta_2\} = (\text{first row of } \mathbf{B}_1^{-1}) \times (\mathbf{a}_1^{(1)}, \mathbf{a}_2^{(1)}) = (1, 0, 0, 0) \begin{bmatrix} -1 & -2 \\ 1 & 1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} = \{-1, -2\}$$

Hence $\Delta_1 = -1$, $\Delta_2 = -2$. Since Δ_1 and Δ_2 both are negative, the solution $x_3 = 3, x_4 = 5, x_5 = 6, z = 0$ is not optimal. Therefore, we proceed to obtain the next improved solution.

Step 2. ***Determination of entering vector $\mathbf{a}_k^{(1)}$.***

To find the entering vector $\mathbf{a}_k^{(1)}$, apply the rule : $\Delta_k = \min [\Delta_1, \Delta_2] = \min [-1, -2] = -2 = \Delta_2$

Hence $k = 2$. So the vector $\mathbf{a}_2^{(1)}$ must enter the basis. This shows that x_2 will enter the basic feasible solution.

Step 3. ***Determination of the leaving vector $\beta_r^{(1)}$, given the entering vector $\mathbf{a}_2^{(1)}$.***

Compute the column $\mathbf{X}_2^{(1)}$ corresponding to vector $\mathbf{a}_2^{(1)}$.

$$\mathbf{X}_2^{(1)} = \mathbf{B}_1^{-1}\, \mathbf{a}_2^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Apply the minimum ratio rule by increasing one more column in *Table 6·7*. This rule shows that [2] is the 'key element' corresponding to which $\beta_2^{(1)}$ must leave the basis matrix. Hence x_4 will be the outgoing variable.

Step 4. ***Determination of the improved solution.***

From *Table 6·7*, the intermediate coefficient matrix is :

$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_2^{(1)}$
0	0	0	0	−2
1	0	0	3	1
0	1	0	5	$\boxed{2}$
0	0	1	6	1
	↓		↑	

Apply usual rules of transformation to obtain

0	2	0	5	0
1	$-1/2$	0	$1/2$	0
0	$1/2$	0	$5/2$	1
0	$-1/2$	1	$7/2$	0

and then construct *Table 6·8* for improved solution.

Table 6·8

Variables in the basis	B_1^{-1}				$X_B^{(1)}$	$X_k^{(1)}$
	e_1	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$		
z	1	0	1	0	5	
$x_3 = x_{B1}$	0	1	$-1/2$	0	$1/2$	
$x_2 = x_{B2}$	0	0	$1/2$	0	$5/2$	
$x_5 = x_{B3}$	0	0	$-1/2$	1	$7/2$	

Additional Table

$a_1^{(1)}$	$a_4^{(1)}$
-1	0
1	0
1	1
3	0

The improved solution now becomes : $z = 5$, $x_3 = 1/2$, $x_2 = 5/2$, $x_5 = 7/2$.

Second Iteration

Step 5. ***Computations of Δ_j for $a_1^{(1)}$ and $a_4^{(1)}$, i.e.,***

$$(\Delta_1 , \Delta_4) = (1 , 0 , 1 , 0) \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \{0 , 1\}$$

Hence $\Delta_1 = 0$, $\Delta_4 = 1$. Since Δ_1 and Δ_4 both are ≥ 0, the solution under test is optimal.

Furthermore, $\Delta_1 = 0$ shows that the problem has alternative optimum solutions. Thus, the required optimal solution is $x_1 = 0, x_2 = 5/2$, max $z = 5$.

Alternative solution can also be obtained as $x_1 = 1$, $x_2 = 2$, max. $z = 5$.

Example 3. *Solve by revised simplex method :*

Max. $z = 6x_1 - 2x_2 + 3x_3$ *subject to* $2x_1 - x_2 + 2x_3 \leq 2, x_1 + 4x_3 \leq 4$ and $x_1 , x_2 , x_3 \geq 0$.

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Solution. The given problem in the revised simplex form may be expressed by introducing the slack variables x_4 and x_5 as

$$\begin{aligned} z - 6x_1 + 2x_2 - 3x_3 \qquad\qquad &= 0 \\ 2x_1 - x_2 + 2x_3 + x_4 \qquad &= 2 \\ x_1 \qquad + 4x_3 \qquad + x_5 &= 4. \end{aligned}$$

The system of constraint equations may be represented in the following matrix form :

$$\begin{array}{cccccc} e_1 & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & a_5^{(1)} \\ \beta_0^{(1)} & & & & \beta_1^{(1)} & \beta_2^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -6 & 2 & -3 & 0 & 0 \\ 0 & 2 & -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

The starting revised simplex table is given below in *Table 6·9*.

Table 6·9

Variables in the Basis	e_1	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)} = \mathbf{X}_1^{(1)}$	Min. $(\mathbf{X}_B/\mathbf{X}_1)$
z	1	0	0	0	-6	↓
$x_4 = x_{B1}$	0	1	0	2	← $\boxed{2}$	2/2 ←
$x_5 = x_{B2}$	0	0	1	4	1 ↑	4/1

$\underbrace{\qquad\qquad}_{\mathbf{B}_1^{-1}}$ (columns e_1, $\beta_1^{(1)}$, $\beta_2^{(1)}$)

Additional Table

$\mathbf{a}_1^{(1)}$	$\mathbf{a}_2^{(1)}$	$\mathbf{a}_3^{(1)}$
-6	2	-3
2	-1	2
1	0	4

The starting solution is : $x_1 = x_2 = x_3 = 0$, $x_4 = 2$, $x_5 = 4$, $z = 0$.

First Iteration

Step 1. ***Computations of*** Δ_j ***for*** $\mathbf{a}_1^{(1)}$, $\mathbf{a}_2^{(1)}$ ***and*** $\mathbf{a}_3^{(1)}$, ***i.e.,*** $(\Delta_1 , \Delta_2 , \Delta_3)$.

$$\{\Delta_1 , \Delta_2 , \Delta_3\} = (\text{first row of } \mathbf{B}_1^{-1})\,(\mathbf{a}_1^{(1)} , \mathbf{a}_2^{(1)} , \mathbf{a}_3^{(1)}) = (1 , 0 , 0)\begin{bmatrix} -6 & 2 & -3 \\ 2 & -1 & 2 \\ 1 & 0 & 4 \end{bmatrix} = \{-6 , 2 , -3\}$$

Hence $\Delta_1 = -6$, $\Delta_2 = 2$, $\Delta_3 = -3$.

Since Δ_1 and Δ_3 are negative, the solution under test is not optimal.

Step 2. ***Determination of the entering vector*** $\mathbf{a}_k^{(1)}$.

The entering vector $\mathbf{a}_k^{(1)}$ corresponds to the value of k which is obtained by the criterion :

$$\Delta_k = \text{min. } [\Delta_1 , \Delta_2 , \Delta_3] = \text{min } \{-6 , 2 , -3\} = -6 = \Delta_1 .$$

Hence $k = 1$.

So the entering vector is found to be $\mathbf{a}_1^{(1)}$. This also means that the variable x_1 will enter the basic solution.

Step 3. ***Determination of the leaving vector*** $\beta_r^{(1)}$ ***, given the entering vector*** $\mathbf{a}_1^{(1)}$.

First we need to compute the column $\mathbf{X}_1^{(1)}$ corresponding to the entering vector $\mathbf{a}_1^{(1)}$.

$$\mathbf{X}_1^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{a}_1^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix}$$

Now apply the min. ratio rule by increasing one more column in Table 6·9. This rule indicates that [2] is the 'key element' corresponding to which $\beta_1^{(1)}$ must leave the basis matrix. Hence x_4 will be the outgoing variable.

Step 4. ***Determination of the first improved solution.***

$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_1^{(1)}$
0	0	0	-6
1	0	2	$\boxed{2}$
0	1	4	1

To transform the *Table 6·9* , transform the above intermediate coefficient matrix.

Apply usual rules of matrix transformation to obtain

3	0	6	0
1/2	0	1	1
$-1/2$	1	3	0

Now construct the transformed Table 6·10 for second iteration.

Table 6·10

Variables in the Basis	$\mathbf{B}_1^{-1}$: e_1	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)}=\mathbf{X}_2^{(1)}$	Min. $(\mathbf{X}_B/\mathbf{X}_2)$ ↓
z	1	3	0	6	-1	
$x_1=x_{B1}$	0	1/2	0	1	$-1/2$	—
$x_5=x_{B2}$	0	$-1/2$	1 ↓	3	$\boxed{1/2}$	$3/\frac{1}{2}$ ←

Additional Table

$\mathbf{a}_4^{(1)}$	$\mathbf{a}_2^{(1)}$	$\mathbf{a}_3^{(1)}$
0	2	-3
1	-1	2
0	0	4

The improved solution is : $z=6$, $x_1=1$, $x_2=x_3=x_4=0$, $x_5=3$.

Second Iteration

Step 5. ***Computations of Δ_j for $\mathbf{a}_4^{(1)}$, $\mathbf{a}_2^{(1)}$, and $\mathbf{a}_3^{(1)}$ (i.e., Δ_4 , Δ_2 , Δ_3) .***

$$\{\Delta_4 , \Delta_2 , \Delta_3\} = (\text{first row of } \mathbf{B}_1^{-1})\ (\mathbf{a}_4^{(1)} , \mathbf{a}_2^{(1)} , \mathbf{a}_3^{(1)}) = (1 , 3 , 0)\begin{bmatrix} 0 & 2 & -3 \\ 1 & -1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \{3 , -1 , 3\}$$

Hence $\Delta_4=3$, $\Delta_2=-1$, $\Delta_3=3$. Since Δ_2 is still negative, the solution under test is not optimal. Hence further improvement is possible. So we proceed to find the '*entering*' and '*leaving*' vectors in the next step.

Step 6. ***Determination of the entering vector $\mathbf{a}_k^{(1)}$.***

Here, we have $\Delta_k = \min. [\Delta_4 , \Delta_2 , \Delta_3] = \min. [3 , -1 , 3] = -1 = \Delta_2$. Hence $k=2$.

Therefore, $\mathbf{a}_2^{(1)}$ must enter the basis. The entering vector $\mathbf{a}_2^{(1)}$ indicates that the variable x_2 must enter the new solution.

Step 7. ***Determination of the leaving vector $\beta_r^{(1)}$, given the entering vector $\mathbf{a}_2^{(1)}$.***

First compute the column $\mathbf{X}_2^{(1)}$ corresponding to vector $\mathbf{a}_2^{(1)}$.

$$\mathbf{X}_2^{(1)} = \mathbf{B}_1^{-1}\mathbf{a}_2^{(1)} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1/2 \\ 1/2 \end{bmatrix}$$

Now complete the column $\mathbf{X}_k^{(1)} = \mathbf{X}_2^{(1)}$ of Table 6·10.

The *'min ratio rule'* in the additional column of *Table 6·10* indicates that 1/2 is the key element corresponding to which the vector $\beta_2^{(1)}$ must leave the basis. Hence x_5 will be the outgoing variable.

Step 8. ***Determination of the next improved solution.***

Transform the Table 6·10 into Table 6·11 from which the next improved solution can be easily read.

Table 6·11

Variables in the Basis	$\mathbf{B}_1^{-1}$: e_1	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)}$
z	1	2	2	12	
$x_1=x_{B1}$	0	0	1	4	
$x_2=x_{B2}$	0	-1	2	6	

Additional Table

$\mathbf{a}_4^{(1)}$	$\mathbf{a}_5^{(1)}$	$\mathbf{a}_3^{(1)}$
0	0	-3
1	0	2
0	1	4

The next improved solution from *Table 6·11* is : $z=12$, $x_1=4$, $x_2=6$, $x_3=x_4=x_5=0$.

Third Iteration

Step 9. ***Computations of Δ_j for $\mathbf{a}_4^{(1)}$, $\mathbf{a}_5^{(1)}$ and $\mathbf{a}_3^{(1)}$, i.e. (Δ_4 , Δ_5 , Δ_3) .***

$$\{\Delta_4 , \Delta_5 , \Delta_3\} = (\text{first row of } \mathbf{B}_1^{-1})\ (\mathbf{a}_4^{(1)} , \mathbf{a}_5^{(1)} , \mathbf{a}_3^{(1)}) = (1 , 2 , 2)\begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} = \{2 , 2 , 9\}$$

Hence $\Delta_4=2$, $\Delta_5=2$, $\Delta_3=9$.

The solution under test is optimal because Δ_4 , Δ_5 , Δ_3 are all positive. Thus, the required optimal solution is :

$$x_1 = 4 \, , x_2 = 6 \, , x_3 = 0 \, , \text{ max. } z = 12 \, . \quad \textbf{Ans.}$$

Example 4. *Solve the following L.P.P. by revised simplex method.*

$$Max \quad z = 3x_1 + x_2 + 2x_3 + 7x_4 \, , \text{ subject to the constraints :}$$

$$2x_1 + 3x_2 - x_3 + 4x_4 \le 40, - 2x_1 + 2x_2 + 5x_3 - x_4 \le 35, x_1 + x_2 - 2x_3 + 3x_4 \le 100, \textit{and}$$

$$x_1 \ge 2 \, , x_2 \ge 1 \, , x_3 \ge 3 \, , x_4 \ge 4 \, .$$

Solution. Step 1. In order to make the lower bounds of the variables zero, we substitute $x_1 = y_1 + 2$, $x_2 = y_2 + 1$, $x_3 = y_3 + 3$, $x_4 = y_4 + 4$ in the given LPP and obtain the following modified problem :

$$\text{Maximize} \quad z' = 3y_1 + y_2 + 2y_3 + 7y_4, \text{ where } z' = z - 41$$

$$\begin{aligned} \text{subject to} \quad & 2y_1 + 3y_2 - y_3 + 4y_4 \le 20 \\ & - 2y_1 + 2y_2 + 5y_3 - y_4 \le 26 \\ & y_1 + y_2 - 2y_3 + 3y_4 \le 91 \end{aligned}$$

and $\quad y_1 \ge 0 \, , y_2 \ge 0 \, , y_3 \ge 0 \, , y_4 \ge 0 \, .$

Step 2. ***To express the modified LPP in revised simplex form.***

$$\text{Max. } z' = 3y_1 + y_2 + 2y_3 + 7y_4, \text{ subject to}$$

$$\begin{aligned} z' - 3y_1 - y_2 - 2y_3 - 7y_4 \qquad\qquad\qquad &= 0 \\ 2y_1 + 3y_2 - y_3 + 4y_4 + y_5 \qquad\qquad &= 20 \\ - 2y_1 + 2y_2 + 5y_3 - y_4 \qquad + y_6 \qquad &= 26 \\ y_1 + y_2 - 2y_3 + 3y_4 \qquad\qquad + y_7 &= 91, \end{aligned}$$

$$y_i \ge 0 \; (i = 1 \, , 2 \, , \ldots , 7), \text{ and } z' \text{ is unrerstricted in sign.}$$

Clearly, the problem is of *standard form–I.*

In matrix form, the system of constraint equations can be written as :

$$\begin{array}{cccccccc} \beta_0^{(1)} & & & & & \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} \\ \mathbf{e}_1 & \mathbf{a}_1^{(1)} & \mathbf{a}_2^{(1)} & \mathbf{a}_3^{(1)} & \mathbf{a}_4^{(1)} & \mathbf{a}_5^{(1)} & \mathbf{a}_6^{(1)} & \mathbf{a}_7^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -3 & -1 & -2 & -7 & 0 & 0 & 0 \\ 0 & 2 & 3 & -1 & 4 & 1 & 0 & 0 \\ 0 & -2 & 2 & 5 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z' \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \\ 26 \\ 91 \end{bmatrix}$$

Step 3. ***To find initial basic solution and the basis matrix*** $\mathbf{B}_1$.

Here $\mathbf{x}_B^{(1)} = (0 \, , 20 \, , 26 \, , 91)$ is the initial BFS and basis matrix $\mathbf{B}_1$ is given by $\mathbf{B}_1 = [\, \beta_0^{(1)} \, , \beta_1^{(1)} \, , \beta_2^{(1)} \, , \beta_3^{(1)}] = \mathbf{I}_4$ (unit matrix). So $\mathbf{B}_1^{-1} = \boldsymbol{I}_4$.

Step 4. ***To construct the starting simplex table.***

Table 6.12

Variables in the basis	$\mathbf{B}_1^{-1}$: $\beta_0^{(1)}$ $\mathbf{e}_1$	$\beta_1^{(1)}$ $\mathbf{a}_5^{(1)}$	$\beta_2^{(1)}$ $\mathbf{a}_6^{(1)}$	$\beta_3^{(1)}$ $\mathbf{a}_7^{(1)}$	Solution $\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)} = \mathbf{X}_4^{(1)} = \mathbf{B}_1^{-1} \mathbf{a}_4^{(1)}$	Min. Ratio $(\mathbf{X}_B/\mathbf{X}_4)$
z'	1	0	0	0	0	-7	
y_5	0	1	0	0	20	$\boxed{4}$	5 ← (min.)
y_6	0	0	1	0	26	-1	—
y_7	0	0	0	1	91	3	91/3
		↓ Outgoing vector				↑ Incoming vector	

Step 5. ***Test for optimality.*** Compute Δ_j for all $\mathbf{a}_j^{(1)}$, $j = 1 \, , 2 \, , 3 \, , 4$ not in the basis.

$$(\Delta_1 \, , \Delta_2 \, , \Delta_3 \, , \Delta_4) = (\text{first row of } \mathbf{B}_1^{-1}) \; [\mathbf{a}_1^{(1)} \, , \mathbf{a}_2^{(1)} \, , \mathbf{a}_3^{(1)} \, , \mathbf{a}_4^{(1)}]$$

$$= (1, 0, 0, 0)\begin{bmatrix} -3 & -1 & -2 & -7 \\ 2 & 3 & -1 & 4 \\ -2 & 2 & 5 & -1 \\ 1 & 1 & -2 & 3 \end{bmatrix} = (-3, -1, -2, -7)$$

Since all Δ_j's are ≤ 0, the solution is not optimal.

Step 6. ***To find incoming and outgoing vectors.***

Incoming vector. $\Delta_k = \min_j \Delta_j = -7 = \Delta_4$, $\therefore k = 4$,

Thus $\mathbf{a}_4^{(1)}$ is the vector entering the basis. So the column vector $\mathbf{X}_4^{(1)}$ coresponding to $\mathbf{a}_4^{(1)}$ is given by

$$\mathbf{X}_4^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{a}_4^{(1)} = \mathbf{I}_4(-7, 4, -1, 3) = [-7, 4, -1, 3]$$

Outgoing vector. Since $\dfrac{x_{Br}}{x_{r4}} = \min\left[\dfrac{20}{4}, -, \dfrac{91}{3}\right] = \dfrac{20}{4} = \dfrac{x_{B1}}{x_{14}}$, so $r = 1$ and hence $\beta_1^{(1)} = a_5^{(1)}$ is the outgoing vector.

$\therefore$ Key element $= x_{14} = 4$, by min. ratio rule.

Step 7. ***To find the improved solution.***

In order to bring $\mathbf{a}_4^{(1)}$ in place of $\beta_1^{(1)}$ ($= \mathbf{a}_5^{(1)}$) in $\mathbf{B}_1^{-1}$, we divide second row by 4 and then add 7, 1 and -3 times in first, third and fourth rows, respectively to get the revised simplex *Table 6·13*.

Table 6·13

Variables in the basis	$\mathbf{B}_1^{-1}$				Solution $\mathbf{X}_B^{(1)}$	$\mathbf{X}_k^{(1)} = \mathbf{X}_3^{(1)}$ $= \mathbf{B}_1^{-1}\mathbf{a}_3^{(1)}$	Min. Ratio $\mathbf{X}_B/\mathbf{X}_3$
	$\beta_0^{(1)}$ $\mathbf{e}_1$	$\beta_1^{(1)}$ $\mathbf{a}_4^{(1)}$	$\beta_2^{(1)}$ $\mathbf{a}_6^{(1)}$	$\beta_3^{(1)}$ $\mathbf{a}_7^{(1)}$			
z'	1	7/4	0	0	35	$-15/4$	
y_4	0	1/4	0	0	5	$-1/4$	—
y_6	0	1/4	1	0	31	$\boxed{19/4}$	124/19 ←
y_7	0	$-3/4$	0	1	76	$-5/4$	—

↓ Outgoing vector ↑ Incoming vector

Step 8. ***Test of optimality for the revised solution Table*** **6·13**.

We compute $(\Delta_1, \Delta_2, \Delta_3, \Delta_5) =$ (first row of $\mathbf{B}_1^{-1}$) $(\mathbf{a}_1^{(1)}, \mathbf{a}_2^{(1)}, \mathbf{a}_3^{(1)}, \mathbf{a}_5^{(1)})$.

$$= (1, 7/4, 0, 0)\begin{bmatrix} -3 & -1 & -2 & 0 \\ 2 & 3 & -1 & 1 \\ -2 & 2 & 5 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} = [1/2, 17/4, -15/4, 7/4]$$

Since $\Delta_3 = -15/4$ is still negative, the solution under test is not optimal. So we proceed to improve the solution in the next step.

Step 9. ***To find entering and outgoing vectors.***

As in ***step 6***, we find the entering vector $\mathbf{a}_3^{(1)}$. The column vector $\mathbf{X}_3^{(1)}$ corresponding to $\mathbf{a}_3^{(1)}$ is given by

$$\mathbf{X}_3^{(1)} = \mathbf{B}_1^{-1}\,\mathbf{a}_3^{(1)} = [-15/4, -1/4, 19/4, -5/4].$$

By *min. ratio rule*, we find the outgoing vector is $\beta_2^{(1)} = \mathbf{a}_6^{(1)}$. So the key element will be 19/4.

Step 10. ***To find the revised solution.***

In order to bring $\mathbf{a}_3^{(1)}$ in place of $\beta_2^{(1)}$ ($= \mathbf{a}_6^{(1)}$) in the basis $\mathbf{B}_1^{-1}$, we divide the third row by 19/4 and then add its 15/4, 1/4 and 5/4 times in first, second and fourth rows respectively to obtain the next revised *Table 6·14*.

Table 6·14

Variables in the basis	B_1^{-1}				Solution $X_B^{(1)}$	$X_k^{(1)} = X_1^{(1)} = B_1^{-1} a_1^{(1)}$	Min ratio X_B/X_1
	$\beta_0^{(1)}$ e_1	$\beta_1^{(1)}$ $a_4^{(1)}$	$\beta_2^{(1)}$ $a_3^{(1)}$	$\beta_3^{(1)}$ $a_7^{(1)}$			
z'	1	37/19	15/19	0	1130/19	– 13/19	
y_4	0	5/19	1/19	0	126/19	$\boxed{8/19}$	63/4 ←
y_3	0	1/19	4/19	0	124/19	– 6/19	—
y_7	0	– 13/19	5/19	1	1599/19	– 17/19	—

↓ Outgoing vector ↑ Incoming vector

Step 11. ***To test the optimality for the revised solution Table 6·14.***

We compute, $[\Delta_1, \Delta_2, \Delta_5, \Delta_6]$ = (first row of B_1^{-1}) $[a_1^{(1)}, a_2^{(1)}, a_5^{(1)}, a_6^{(1)}]$

$$= \left[1, \frac{37}{19}, \frac{15}{19}, 0\right] \begin{bmatrix} -3 & -1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ -2 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \left[\frac{-13}{19}, \frac{122}{19}, \frac{37}{19}, \frac{15}{19}\right]$$

Since $\Delta_1 < 0$, the solution under test is not optimal. So we proceed to revise the solution in the next step.

Step 12. ***To find entering and outgoing vectors.***

As in ***step 6,*** we find the entering vector $a_1^{(1)}$. The column vector corresponding to $a_1^{(1)}$ is given by

$$X_1^{(1)} = B_1^{-1} a_1^{(1)} = \left[\frac{-13}{19}, \frac{8}{18}, \frac{-6}{19}, \frac{-17}{19}\right]$$

By **min ratio rule,** we find the outgoing vector is $\beta_1^{(1)} = a_4^{(1)}$. So the key element is 8/19.

Step 13. ***To find the improved solution.***

In order to bring $a_1^{(1)}$ in place of $\beta_1^{(1)} (= a_4^{(1)})$, we divide second row by 8/19, then add its 13/19, 6/19 and 17/19 times in first, third and fourth rows respectively to obtain the next improved solution Table 6·15.

Table 6·15

Varaibles in the basis	B_1^{-1}				Solution $X_B^{(1)}$
	$\beta_0^{(1)}$ e_1	$\beta_1^{(1)}$ $a_1^{(1)}$	$\beta_2^{(1)}$ $a_3^{(1)}$	$\beta_3^{(1)}$ $a_7^{(1)}$	
z'	1	19/8	7/8	0	281/4
y_1	0	5/8	1/8	0	63/4
y_3	0	1/4	1/4	0	23/2
y_7	0	– 1/8	3/8	1	393/4

Step 14. ***To test the optimality of the improved solution Table 6·15.***

We compute, $(\Delta_2, \Delta_4, \Delta_5, \Delta_6)$ = (first row of B_1^{-1}) $(a_2^{(1)}, a_4^{(1)}, a_5^{(1)}, a_6^{(1)})$

$$= \left(1, \frac{19}{8}, \frac{7}{8}, 0\right) \begin{bmatrix} -1 & 7 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 2 & -1 & 0 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix} = \left(\frac{63}{8}, \frac{13}{8}, \frac{19}{8}, \frac{7}{8}\right)$$

Since all $\Delta_j > 0$, the solution under test is optimal. So the optimal solution of modified LPP is,

$$y_1 = 63/4, y_2 = 0, y_3 = 23/2, y_4 = 0 \text{ and } \max z' = 281/4.$$

Transforming this solution for the original LPP, we get the desired solution as,

$x_1 = y_1 + 2 = 71/4$, $x_2 = y_2 + 1 = 1$, $x_3 = y_3 + 3 = 29/2$, $x_4 = y_4 + 4 = 4$

and $\max z = \max (z' + 41) = 445/4$. **Ans.**

6.8. SUMMARY OF REVISED SIMPLEX METHOD IN STANDARD FORM–I (Computational Procedure) [Meerut 90]

The computational procedure of revised simplex method in *standard form*-I (when no artificial variables are needed) may be more conveniently out-lined as follows :

Step 1. ***If the problem is of minimization; convert it into the maximization problem.***

Step 2. Express the given problem in Standard Form–I.

After ensuring that all $b_i \geq 0$, express the given problem in revised simplex form-I as explained in **section** 6·3.

Step 3. Find the initial basic feasible solution and the basis matrix $\mathbf{B_1}$.

In this step, we proceed to obtain the initial basis matrix $\mathbf{B_1}$ as an identity matrix. Thus the initial solution is given by $\mathbf{x_B}^{(1)} = (0 , b_1 , b_2 , \ldots, b_m)$.

Step 4. Construct the starting table for revised simplex method as explained in section 6·6.

Step 5. Test the optimality of current BFS.

This is done by computing $\Delta_j = z_j - c_j$ for all $\mathbf{a_j}^{(1)}$ not in the basis $\mathbf{B_1}$ by the formula :

$$\Delta_j = (\text{first row of } \mathbf{B_1}^{-1}) \times (\mathbf{a_j}^{(1)} \text{ not in this basis})$$

The BFS is optimal only when all $\Delta_j \geq 0$.

If current BFS is neither optimal nor unbounded, proceed to improve it in the next step.

Step 6. Improve the BFS.

In this step, we first find the ***incoming*** (entering) vector and the ***leaving*** (outgoing) vector to obtain the key element. Then we determine the improved solution like regular simplex method as follows :

(i) ***To find in-coming vector.*** The incoming vector will be taken as $\mathbf{a_k}^{(1)}$ if $\Delta_k = \min_j (\Delta_j)$ for those j for which $\mathbf{a_j}^{(1)}$ are not in the basis $\mathbf{B_1}$.

(ii) ***To find out-going vector.*** For this, first we compute $\mathbf{X_k}^{(1)}$ by the formula :

$$\mathbf{X_k}^{(1)} = \mathbf{B_1}^{-1}\, \mathbf{a_k}^{(1)} = [\Delta_k , x_{1k} , x_{2k} , \ldots , x_{mk}]$$

The vector $\beta_r^{(1)}$ to be removed from the basis is determined by using the *minimum ratio rule.* That is, it is selected corresponding to such value of r for which

$$\frac{x_{Br}}{x_{rk}} = \min_i \left[\frac{x_{Bi}}{x_{ik}} , x_{ik} > 0\right]$$

Note. Here $\mathbf{a_k}^{(1)}$ is the in-coming vector and $\mathbf{X_k}^{(1)}$ is the column vector corresponding to $\mathbf{a_k}^{(1)}$.

(iii) ***To find the key element.*** When $\mathbf{a_k}^{(1)}$ is the in-coming vector and $\beta_r^{(1)}$ is the out-going vector, the ***key-element*** x_{rk} is situated at the intersection of rth row and kth column of the matrix.

(iv) ***To transform the revised simplex table.***

In order to bring $\mathbf{a_k}^{(1)}$ in place of $\beta_r^{(1)}$, we proceed similarly as in ordinary simplex method and then construct the new (revised) simplex table.

In this manner, we obtain the improved BFS.

Step 7. Now again test the optimality of above improved BFS as in *Step 5*

If this solution is not optimal, then repeat ***step 6*** until an optimal solution is finally obtained.

Q. Give a brief outline for the standard form I of the revised simplex method. **[Delhi BSc. (Maths) 93, 91, 90]**

EXAMINATION PROBLEMS

Use revised simplex method to solve the following linear programming problems :

1. Max. $z = x_1 + x_2$
subject to the constraints :
$3x_1 + 3x_2 \leq 6$
$x_1 + 4x_2 \leq 4$
$x_1 , x_2 > 0$.
[Ans. $x_1 = \frac{8}{5}$, $x_2 = \frac{3}{5}$, max. $z = \frac{11}{5}$]

2. Max. $z = x_1 + 2x_2$
subject to
$x_1 + 2x_2 \leq 3$
$x_1 + 3x_2 \leq 1$
$x_1 , x_2 \leq 0$.
[Ans. $x_1 = 1$, $x_2 = 0$, $z^* = 1$]

3. Max. $z = 5x_1 + 3x_2$
subject to
$3x_1 + 5x_2 \leq 15$
$3x_1 + 2x_2 \leq 10$
$x_1 , x_2 \geq 0$.
[Ans. $x_1 = \frac{22}{19}$, $x_2 = \frac{45}{19}$, $z^* = \frac{285}{19}$]

4. Max. $z = 3x_1 + 2x_2 + 5x_3$
subject to
$x_1 + 2x_2 + x_3 \le 430$
$3x_1 + 2x_3 \le 460$
$x_1 + 4x_2 \le 420$
$x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 0$, $x_2 = 100$, $x_3 = 230$, $z^* = 1350$]

5. Max. $z = x_1 + x_2 + 3x_3$
subject to the constraints :
$3x_1 + 2x_2 + x_3 \le 3$
$2x_1 + x_2 + 2x_3 \le 2$
$x_1, x_2, x_3 \ge 0$.
[**Ans.** $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $z'' = 3$]

6. Max. $z = 30x_1 + 23x_2 + 29x_3$
subject to the constraints :
$6x_1 + 5x_2 + 3x_3 \le 26$
$4x_1 + 2x_2 + 5x_3 \le 7$
and $x_1, x_2, x_2 \ge 0$
[Meerut M.A. (P) 93]
[**Ans.** $x_1 = 0$, $x_2 = 7/2$, $x_3 = 0$, $z^* = 161/2$]

7. Max. $z = x_1 + x_2$,
s.t. $x_1 + 2x_2 \le 2$
$4x_1 + x_2 \le 4$
$x_1, x_2 \ge 0$
[**Ans.** $x = 6/7$, $x_2 = 4/7$, max $z = 10/7$]

8. Max. $z = 2x_1 + 3x_2$
s.t. $x_2 - x_1 \ge 0$, $x_1 \le 4$, and
$x_1, x_2 \ge 0$
[**Ans.** Unbounded sol.]

9. Explain the revised simplex method and compare it with the **simplex** method.

Revised Simplex Method in Standard Form–II

6.9. FORMULATION OF LPP IN STANDARD FORM–II

The *Standard Form –II* is used when artificial variables are required to obtain the initial basis matrix as an identity matrix. The unit vectors correspond to either *slack* or *artificial* variables. If the revised simplex procedure starts with artificial vectors, there will be a problem of driving all artificial variables to zero to obtain an initial feasible solution (if exists). Thereafter, iterative procedure for successive improved solutions may be continued till the optimality conditions are satisfied. *Two-Phase-Method* will now be used in a slightly different manner.

In Phase I, an initial basic feasible solution is found by driving all artificial variables to zero. While, in Phase II, we start with the initial basic feasible solution obtained in Phase I. While discussing the *Standard Form I* in the preceding sections, it is found very useful to treat the objective function as one of the constraint equations.

Now the approach in *Phase I* and *Phase II* is discussed.

Phase I.[Initial basic feasible solution driving all artificial variables to zero].

Suppose each constraint of the problem is present in the form of equation (although no difficulty will arise even if some or all of the constraints are inequalities). As explained earlier, one artificial variable is required to be introduced in each of the m constraint equations in order to get the basis matrix as an identity matrix. Thus, for the revised simplex method, the L.P. problem can be written as follows:

Maximize z, subject to the constraint

$$\left.\begin{array}{llllll} z - c_1x_1 - c_2x_2 - \dots - c_nx_n & & & & & = 0 \\ a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + x_{n+1} & & & & & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & & + x_{n+2} & & & = b_2 \\ \vdots \quad\quad \vdots \quad\quad \vdots & & \vdots & & & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & & & & + x_{n+m} & = b_m \end{array}\right\} \quad \dots(6{\cdot}22)$$

Since it is desirable to introduce one more additional constraint equation (its utility and method of construction will be explained just now) at the second place of constraint system (6·22), it will be convenient to leave space at the second place shown by dotted line (...........). Thus, for convenience, the constraint equations (6·22) are re-written as follows :

$$\left.\begin{array}{llllll} z - c_1x_1 - c_2x_2 - \dots - c_nx_n & & & & & = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & & & & & \\ a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + x_{n+1} & & & & & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & & + x_{n+2} & & & = b_2 \\ \vdots \quad\quad \vdots \quad\quad \vdots & & \vdots & & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & & & & + x_{n+m} & = b_m \end{array}\right\} \quad \dots(6{\cdot}22)'$$

To generalize the concept of artificial vectors and facilitate the computation of Phase 1, calculate a *'redundant'* constraint equation which is called the *second constraint equation.* Now the system (6·22) of constraint equations together with the newly calculated second redundant equation becomes

$$\begin{array}{llllll} z-c_1x_1 & -c_2x_2- & \dots -c_nx_n & & =0 & \text{(i)} \\ \bar{a}_{11}x_1 & +\bar{a}_{12}x_2+\dots & +\bar{a}_{1n}x_n+\bar{x}_{n+1} & & =\bar{b}_1 & \text{(ii)} \\ a_{11}x_1 & +a_{12}x_2+\dots & +a_{1n}x_n & +x_{n+1} & =b_1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \text{(iii)} \\ a_{m1}x_1 & +a_{m2}x_2+\dots & +a_{mn}x_n & \qquad +x_{n+m} & =b_m & \end{array}$$

where constraint (i) is the objective function, and the additional constraint (ii) is so formed as :

$$\bar{a}_{11}=-(a_{11}+a_{21}+\dots+a_{m1})=-\sum_{i=1}^{m}a_{i1}$$

$$\bar{a}_{12}=-(a_{12}+a_{22}+\dots+a_{m2})=-\sum_{i=1}^{m}a_{i2}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\bar{a}_{1n}=-(a_{1n}+a_{2n}+\dots+a_{mn})=-\sum_{i=1}^{m}a_{in}$$

and
$$\bar{b}_1=-(b_1+b_2+\dots+b_m)=-\sum_{i=1}^{m}b_i,$$

where $\bar{x}_{n+1}$ is the variable unrestricted in sign. The system [(i), (ii) and (iii)] of constraint equations can be more systematically written as

$$\left.\begin{array}{llll} x_0+a_{01}x_1+a_{02}x_2+\dots+a_{0n}x_n & & & =0 \\ \bar{a}_{11}x_1+\bar{a}_{12}x_2+\dots+\bar{a}_{1n}x_n+\bar{x}_{n+1} & & & =\bar{b}_1 \\ a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n & +x_{n+1} & & =b_1 \\ \vdots & & & \vdots \\ a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n & & +x_{n+m} & =b_m \end{array}\right\} \qquad \dots(6\cdot23)$$

where $z=x_0, -c_j=a_{0j}$, and $j=1, 2, \dots, n$.

Adding all constraint equations except *the first one*, we get

$$\left(\bar{a}_{11}+\sum_{i=1}^{m}a_{i1}\right)x_1+\left(\bar{a}_{12}+\sum_{i=1}^{m}a_{i2}\right)x_2+\dots+\left(\bar{a}_{1n}+\sum_{i=1}^{m}a_{in}\right)x_n+\bar{x}_{n+1}+x_{n+1}\dots+x_{n+m}=\left(\bar{b}_1+\sum_{i=1}^{m}b_i\right) \qquad \dots(6\cdot24)$$

Further, substituting the values of $\bar{a}_{11}, \bar{a}_{12}, \dots, \bar{a}_{1n}$ and $\bar{b}_1$ from above relations, we get

$$\bar{x}_{n+1}+x_{n+1}+\dots+x_{n+m}=0. \qquad \dots(6\cdot24a)$$

Since all the artificial variables $x_{n+1}, x_{n+2}, \dots, x_{n+m}$ are non-negative, $\bar{x}_{n+1}$ can never be positive. So, in *Phase I,* the problem is to maximize $\bar{x}_{n+1}$ [not $x_0=z$] subject to the system of constraint equations, with x_0 and $\bar{x}_{n+1}$ both unrestricted in sign. This process will result as follows :

(a) ***either max.*** $\bar{x}_{n+1}=0$, which automatically drives all artificial variables equal to zero and the original variables x_j for $j=1, 2, \dots, n$ of this "preliminary maximum solution" represent a basic feasible solution to start with *Phase II* for maximizing z or x_0.

(b) **or max $\bar{x}_{n+1}$ is negative,** which clearly shows that at least one artificial variable still has a non-negative value, and hence no feasible solution exists to the original problem.

Phase II. After driving all artificial vectors to zero, enter *Phase II.* The procedure in *Phase II* will be exactly the same as in **section 6·6** for *standard form–I.* Detailed procedure of *Phase II* has been made clear in **section 6·12.**

6.10. NOTATIONS AND BASIS MATRIX IN STANDARD FORM–II

The system of constraint equations (6·23) in the preceding section can be expressed in the matrix form as follows :

$$\begin{bmatrix} 1 & a_{01} & a_{02} & \dots & a_{0n} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \bar{a}_{11} & \bar{a}_{12} & \dots & \bar{a}_{1n} & 1 & 0 & \dots & 0 \\ 0 & a_{11} & a_{12} & \dots & a_{1n} & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & a_{m1} & a_{m2} & \dots & a_{mn} & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \\ \bar{x}_{n+1} \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{b}_1 \\ b_1 \\ \vdots \\ b_m \end{bmatrix} \qquad \dots(6{\cdot}25)$$

Basis. From matrix equation (6·25), the starting basis matrix will be an $(m+2) \times (m+2)$ unit matrix denoted by $\mathbf{I}_{m+2}$. Since the initial basis matrix $\mathbf{I}_{m+2}$ is of order $m+2$ in *standard form–II*, use a subscript $_2$ on $\mathbf{B}$ ($\mathbf{B}$ is the original basis matrix). Thus, the initial basis matrix $\mathbf{B}_2$ *for standard form–II* is given by

$$\begin{array}{cc} & \begin{array}{cccccc} x_0 & \bar{x}_{n+1} & & x_{n+1} & \cdots & x_{n+m} \end{array} \\ \mathbf{B}_2 = & \begin{pmatrix} 1 & 0 & : & 0 & \dots & 0 \\ 0 & 1 & : & 0 & \dots & 0 \\ \dots & \dots & : & \dots & \dots & \dots \\ 0 & 0 & : & 1 & \dots & 0 \\ \vdots & \vdots & : & \vdots & \mathbf{B} & \vdots \\ 0 & 0 & : & 0 & \dots & 1 \end{pmatrix} \end{array}$$

Since $\mathbf{B}_2 = \mathbf{I}_{m+2}$, $\mathbf{B}_2^{-1} = \mathbf{I}_{m+2} = \mathbf{B}_2$ initially. Consequently, last m rows and m columns of $\mathbf{B}_2$ represent the inverse of original initial basis $\mathbf{B}$.

Since all the vectors have $m+2$ components, a superscript $^{(2)}$ on vectors will indicate the quantities appropriate to *standard form–II.* For example, matrix $\mathbf{B}_2$ can be written in terms of vectors as :

$$\mathbf{B}_2 = (\mathbf{e}_1, \ \mathbf{e}_2, \ \beta_1^{(2)}, \ \beta_2^{(2)}, \dots, \beta_m^{(2)})$$

6.11. COMPUTATIONAL PROCEDURE FOR STANDARD FORM–II

The computational steps for *Phase I* and *Phase II* are as follows :

Phase I. ***When artificial variables are present in the initial solution with positive values.***

Step 1. First construct the simplex table in the following form.

Initial Table 6·16

Variables in the basis	$\mathbf{e}_1$	$\mathbf{e}_2$	$\beta_1^{(2)}$	$\beta_2^{(2)}$	...	$\beta_m^{(2)}$	$\mathbf{X}_B^{(2)}$	$\mathbf{X}_k^{(2)}$
x_0	1	0	0	0	...	0		
$\bar{x}_{n+1}$	0	1	0	0		0		
x_{n+1}	0	0	1	0	...	0		
x_{n+2}	0	0	0	1	...	0		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$		
x_{n+m}	0	0	0	0	...	1		

In *Phase I*, maximize $\bar{x}_{n+1}$ (not $x_0 = z$), as explained in the previous section.

Step 2. If $\bar{x}_{n+1} < 0$, compute $\Delta_j \equiv z_j - c_j =$ [second row of $\mathbf{B}_2^{-1}$] $[\mathbf{a}_j^{(2)}]$ and continue to *step 3* and so on.

(ii) If max. $\bar{x}_{n+1} = 0$, go to *Phase II.*

Step 3. ***To find the vector to be introduced into the basis.***

(i) If $\Delta_j \geq 0$, $\bar{x}_{n+1}$ is at its maximum and hence no feasible solution exists for the problem.

(ii) If at least one $\Delta_j < 0$, the vector to be introduced in the basis, $\mathbf{X}_k^{(2)}$, corresponds to such value of k which is obtained by : $\Delta_k = \min \Delta_j$ (for those j for which $\mathbf{a}_j^{(2)}$ not in the basis).

(iii) If more than one value of Δ_j are equal to the maximum, we select Δ_k such that k is the smallest index.

Step 4. ***To compute $\mathbf{X}_k^{(2)}$ by using the formula :*** $\mathbf{X}_k^{(2)} = \mathbf{B}_2^{-1}\, \mathbf{a}_k^{(2)}$,

Step 5. ***To find the vector to be removed from the basis.***

The vector to be removed from the basis is obtained by using the *minimum ratio rule :*

$$\frac{x_{Br}}{x_{rk}} = \min \left\{ \frac{x_{Bi}}{x_{ik}},\, x_{ik} > 0 \right\},\ i = 1, 2, \ldots, m.$$

If there is degeneracy, resolve it by using one of the methods described in *Chapter 3.*

Step 6. After determining the *'incoming'* and *'outgoing'* vectors, next revised simplex table can be easily obtained.

Repeat the procedure of *Phase* I to get max. $\bar{x}_{n+1} = 0$ or all Δ_j for *Phase I* are ≥ 0.

If max. $\bar{x}_{n+1}$ comes out to be zero in Phase I, all artificial variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ must have the value zero. It should be noted carefully that max. $\bar{x}_{n+1}$ will always come out to be zero at the end of *phase I* if the feasible solution to the problem exists.

We now proceed to maximize $x_0\ (= z)$ in *Phase II.*

Phase II. Maximize $x_0\ (= z)$.

In *Phase II*, $\bar{x}_{n+1}$ is considered like any other artificial variable ; it can be removed from the basic solution. Only x_0 must always remain in the basic solution. However, there will always be at least one artificial vector in $\mathbf{B}_2$, otherwise it is not possible to have an $m + 2$-dimensional basis.

The procedure in *Phase II* will be the same as described in ***standard form–I.***

Compute $\Delta_j \equiv z_j - c_j =$ [first row of $\mathbf{B}_2^{-1}$] $[\mathbf{a}_j^{(2)}]$ for those values of j for which $\mathbf{a}_j^{(2)}$ are not in the basis.

Then, find the *'incoming vector'* and *'outgoing vector'* as described in ***standard form–I***, and continue improving the current solution till the optimality conditions of simplex method are satisfied.

Q. 1. Describe the revised simplex method, when artificial vectors are added to obtain the identity matrix for initial basis matrix.
2. Standard form II of the Revised Simplex Method is used for solving an L. P.P. of which type ?
3. Give brief outlines of the standard form II of the revised simplex method.

Example 5. *Solve by revised simplex method the problem :*

Max. $z = x_1 + 2x_2 + 3x_3 - x_4$, *subject to the constraints*

$$x_1 + 2x_2 + 3x_3 = 15,\quad 2x_1 + x_2 + 5x_3 = 20,\quad x_1 + 2x_2 + x_3 + x_4 = 10,\quad \textit{and } x_1, x_2, x_3, x_4 \geq 0.$$

Solution. ***Step 1.*** First write the objective function as first constraint. *i.e.*

$$z - x_1 - 2x_2 - 3x_3 + x_4 = 0.$$

Then introducing the artificial variables x_6, x_7, x_8 to each of the three constraint equations respectively, we get

$$\begin{aligned} x_1 + 2x_2 + 3x_3 \quad\ + x_6 \qquad\quad &= 15 \\ 2x_1 + x_2 + 5x_3 \qquad\quad + x_7 \quad\ &= 20 \\ x_1 + 2x_2 + x_3 + x_4 \qquad\qquad + x_8 &= 10. \end{aligned}$$

Form a new constraint equation taking the *negative sum of above three constraints* which have been given at the second place of the new system of constraint equations. Thus, the second constraint equation now becomes

$$-4x_1 - 5x_2 - 9x_3 - x_4 - (x_6 + x_7 + x_8) = -45$$

or

$$-4x_1 - 5x_2 - 9x_3 - x_4 + x_5 = -45, \quad \text{where} \quad x_5 = -(x_6 + x_7 + x_8).$$

Rewriting the new system of constraint equations in proper form (including the objective function),

$$\begin{aligned} z - x_1 - 2x_2 - 3x_3 + x_4 \qquad\qquad\qquad &= 0 \\ -4x_1 - 5x_2 - 9x_3 - x_4 + x_5 \qquad\qquad &= -45 \\ x_1 + 2x_2 + 3x_3 \qquad + x_6 \qquad\quad &= 15 \\ 2x_1 + x_2 + 5x_3 \qquad\quad + x_7 \quad\ &= 20 \\ x_1 + 2x_2 + x_3 + x_4 \qquad\qquad + x_8 &= 10. \end{aligned}$$

Expressing the system of constraint equations in the matrix form.

$$\begin{array}{ccccccccc} \mathbf{a}_0^{(2)} & \mathbf{a}_1^{(2)} & \mathbf{a}_2^{(2)} & \mathbf{a}_3^{(2)} & \mathbf{a}_4^{(2)} & \mathbf{a}_5^{(2)} & \mathbf{a}_6^{(2)} & \mathbf{a}_7^{(2)} & \mathbf{a}_8^{(2)} \\ (\mathbf{e}_1) & & & & & (\mathbf{e}_2) & (\beta_1^{(2)}) & (\beta_2^{(2)}) & (\beta_3^{(2)}) \end{array}$$

$$\begin{pmatrix} 1 & -1 & -2 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -4 & -5 & -9 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 5 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} 0 \\ -45 \\ 15 \\ 20 \\ 10 \end{pmatrix}$$

Here the basis matrix $\mathbf{B}_2$ (which is the identity matrix) and matrix $\mathbf{A}_2$ are

$$\mathbf{B}_2 = \begin{array}{c} \begin{array}{ccccc} \mathbf{e}_1 & \mathbf{e}_2 & \beta_1^{(2)} & \beta_2^{(2)} & \beta_3^{(2)} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}, \quad \mathbf{A}_2 = \begin{array}{c} \begin{array}{cccc} \mathbf{a}_1^{(2)} & \mathbf{a}_2^{(2)} & \mathbf{a}_3^{(2)} & \mathbf{a}_4^{(2)} \end{array} \\ \begin{pmatrix} -1 & -2 & -3 & 1 \\ -4 & -5 & -9 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 1 & 5 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix} \end{array}$$

In the basis matrix $\mathbf{B}_2$, first two columns (denoted by $\mathbf{e}_1$, $\mathbf{e}_2$) associated with the variables z and x_5 will never change because they are unrestricted in sign (that is, may be *negative, positive* or *zero*).

First enter *Phase I* to maximize x_5 (not z). The phase I will come to an end at the iteration where the maximum value of x_5 comes out to be zero (max. $x_5 = 0$).

Phase I. Construct the starting revised simplex table for Phase I.

Table 6·17

Varables in the basis	$\mathbf{B}_2^{-1}$ e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	$\mathbf{X}_B^{(2)}$	$\mathbf{X}_k^{(2)}$ $(k=3)$	Min. $(\mathbf{X}_B/\mathbf{X}_3)$
z	1	0	0	0	0	0	-3	
x_5	0	1	0	0	0	-45	-9	
x_6	0	0	1	0	0	15	3	15/3
x_7	0	0	0	1	0	20	$\boxed{5}$	20/5 ←
x_8	0	0	0	0	1	10	1	10/1

↓

(Completed in step 3)

Additional Table

$\mathbf{a}_1^{(2)}$	$\mathbf{a}_2^{(2)}$	$\mathbf{a}_3^{(2)}$	$\mathbf{a}_4^{(2)}$
-1	-2	-3	1
-4	-5	-9	-1
1	2	3	0
2	1	5	0
1	2	1	1

First Iteration

Step 1. ***Computations of*** Δ_j **for** $\mathbf{a}_1^{(2)}$, $\mathbf{a}_2^{(2)}$, $\mathbf{a}_3^{(2)}$ **and** $\mathbf{a}_4^{(2)}$, *i.e.* (Δ_1, Δ_2, Δ_3 and Δ_4)

Δ_1 = (second row of $\mathbf{B}_2^{-1}$) $\times$ ($\mathbf{a}_1^{(2)}$) = $(0, 1, 0, 0, 0)(-1, -4, 1, 2, 1) = (0 - 4 + 0 + 0 + 0 + 0) = -4$

Δ_2 = (second row of $\mathbf{B}_2^{-1}$) ($\mathbf{a}_2^{(2)}$) = $(0, 1, 0, 0, 0)(-2, -5, 2, 1, 2) = -5$

$\Delta_3 = (0, 1, 0, 0, 0)(-3, -9, 3, 5, 1) = -9$ and $\Delta_4 = (0, 1, 0, 0, 0)(1, -1, 0, 0, 1) = -1$.

Since Δ_1, Δ_2, Δ_3, Δ_4 are all negative, the solution (in which $x_5 = -45$) can be further improved to maximize x_5.

Step 2. ***Determination of the entering vector*** $\mathbf{a}_k^{(2)}$.

The entering vector $\mathbf{a}_k^{(2)}$ corresponds to such value of k which is obtained by the criterion

$$\Delta_k = \min[\Delta_1, \Delta_2, \Delta_3, \Delta_4] = \min.[-4, -5, -9, -1] = -9 = \Delta_3. \quad \text{Hence } k = 3.$$

The entering vector is thus found to be $\mathbf{a}_3^{(2)}$, thereby means that the variable x_3 will enter the basic solution (x_3 will have a positive value instead of zero).

Step 3. ***Determination of leaving vector*** $\beta_r^{(2)}$ ***given the entering vector*** $\mathbf{a}_3^{(2)}$.

At this stage, compute the column $\mathbf{X}_k^{(2)} = \mathbf{X}_3^{(2)}$ of *Table* 6·17 corresponding to the entering vector $\mathbf{a}_3^{(2)}$,

$$\mathbf{X}_3^{(2)} = \mathbf{B}_2^{-1}\,\mathbf{a}_3^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -9 \\ 3 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -9 \\ 3 \\ 5 \\ 1 \end{bmatrix}$$

Apply the *minimum ratio rule* by adding one more column in the *Table* 6·17 . This rule indicates that $\boxed{5}$ is the 'key element' corresponding to which $\beta_2^{(2)}$ $[\mathbf{a}_7^{(2)}]$ must be removed. Hence x_7 will be the outgoing variable (so x_7 will have the value zero).

Step 4. ***Determination of the first improved value of*** x_5 .

To obtain the transformed basis matrix, write the intermediate coefficient matrix as before and apply usual rules of matrix transformation.

$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	$\mathbf{X}_B^{(2)}$	$\mathbf{X}_3^{(2)}$
0	3/5	0	12	0
0	9/5	0	-9	0
1	− 3/5	0	3	0
0	1/5	0	4	1
0	− 1/5	1	6	0

For second iteration, next table has been formed.

Table 6·18

Variables in the basis	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	$\mathbf{X}_B^{(2)}$	$\mathbf{X}_k^{(2)}$ $(k=2)$	Min. (X_B/X_2)
z	1	0	0	3/5	0	12	− 7/5	
x_5	0	1	0	9/5	0	−9	− 16/5	
$\leftarrow x_6$	0	0	1	− 3/5	0	3	$\boxed{7/5}$	3/(7/5) ←
$\rightarrow x_3$	0	0	0	1/5	0	4	1/5	4/(1/5)
x_8	0	0	0	− 1/5	1	6	9/5	6/(9/5)

↓

To be completed in step 3

Additional Table

$\mathbf{a}_1^{(2)}$	$\mathbf{a}_2^{(2)}$	$\mathbf{a}_7^{(2)}$	$\mathbf{a}_4^{(2)}$
1	−2	0	1
-1	−5	0	−1
1	2	0	0
2	1	1	0
1	2	0	1

In this table $x_5 = -9$ (which is not zero), hence enter the second iteration.

Second Iteration

Step 1. ***Computations of*** Δ_j **for** $\mathbf{a}_1^{(2)}, \mathbf{a}_2^{(2)}, \mathbf{a}_7^{(2)}, \mathbf{a}_4^{(2)}$, **i.e.** $(\Delta_1, \Delta_2, \Delta_7$ **and** $\Delta_4)$

$\Delta_1 =$ (second row of $\mathbf{B}_2^{-1}$) $(\mathbf{a}_1^{(2)}) = (0, 1, 0, 9/5, 0)\,(-1, 4, 1, 2, 1) = -2/5$;

$\Delta_2 = (0, 1, 0, 9/5, 0)\,(-2, -5, 2, 1, 2) = -16/5$; $\Delta_7 = (0, 1, 0, 9/5, 0)\,(0, 0, 0, 1, 0) = 9/5$;

$\Delta_4 = (0, 1, 0, 9/5, 0)\,(1, -1, 0, 0, 1) = -1$.

Since $\Delta_1, \Delta_2, \Delta_4$ are still negative, the present value of x_5 is −9 which is not maximum. Hence proceed for next improvement.

Step 2. ***Determination of the entering vector*** $\mathbf{a}_k^{(2)}$.

The entering vector $\mathbf{a}_k^{(2)}$ corresponds to such value of k which can be obtained by the criterion :

$$\Delta_k = \min\,[\Delta_1, \Delta_2, \Delta_7, \Delta_4] = \min\,[-2/5, -16/5, 9/5, -1] = -16/5 = \Delta_2, \text{ Hence } k = 2.$$

So the vector to be introduced into the basis is $\mathbf{a}_2^{(2)}$. Hence the variable x_2 will enter the basic solution.

Step 3. ***Determination of the leaving vector*** $\beta_r^{(2)}$ ***, given the entering vector*** $\mathbf{a}_2^{(2)}$.

Compute

$$\mathbf{X}_2^{(2)} = \mathbf{B}_2^{-1}\,\mathbf{a}_2^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 3/5 & 0 \\ 0 & 1 & 0 & 9/5 & 0 \\ 0 & 0 & 1 & -3/5 & 0 \\ 0 & 0 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & -1/5 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7/3 \\ -16/5 \\ 7/5 \\ 1/5 \\ 9/5 \end{bmatrix}$$

Now it is possible to complete the column $\mathbf{x}_k^{(2)} = \mathbf{x}_2^{(2)}$ of *Table 6·18*.

To find the leaving vector $\beta_r^{(2)}$, add one more column for min. ratio rule. It has been found that 7/5 will be the *key element,* consequently $\beta_1^{(2)}$ must be removed from the basis.

Step 4. ***Determination of next improved value of*** $\mathbf{x}_5$.

To get the transformed basis matrix, proceed in the usual manner. First, write the intermediate coefficient matrix as

$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	$\mathbf{x}_B^{(2)}$	$\mathbf{x}_2^{(2)}$
0	3/5	0	12	−7/5
0	9/5	0	−9	−16/5
1	−3/5	0	3	$\boxed{7/5}$
0	1/5	0	4	1/5
0	−1/5	1	6	9/5

Then, by usual matrix transformation rules,

1	0	0	15	0
16/7	3/7	0	−15/7	0
5/7	−3/7	0	15/7	1
−1/7	2/7	0	25/7	0
−9/7	4/7	1	15/7	0

Construct Table 6·19 for third iteraion.

Table 6·19

Variables in the basis	$\mathbf{B}_2^{-1}$					$\mathbf{X}_B^{(2)}$	$\mathbf{X}_k^{(2)}$	
	$\mathbf{e}_1$	$\mathbf{e}_2$	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$			**Min. Ratio** $(\mathbf{X}_B/\mathbf{X}_4)$
z	1	0	1	0	0	15	1	
x_5	0	1	16/7	3/7	0	−15/7	−1	
$\rightarrow x_2$	0	0	5/7	−3/7	0	15/7	0	...
x_3	0	0	−1/7	2/7	0	25/7	0	...
$\leftarrow x_8$	0	0	−9/7	4/7	1	15/7	1	15/7 ←
					↓		↑	

Additional Table

$\mathbf{a}_1^{(2)}$	$\mathbf{a}_6^{(2)}$	$\mathbf{a}_7^{(2)}$	$\mathbf{a}_4^{(2)}$
−1	0	0	1
−4	0	0	−1
1	1	0	0
2	0	1	0
1	0	0	1

Third Iteration

Step 1. ***Computations of*** Δ_j ***for*** $\mathbf{a}_1^{(2)}, \mathbf{a}_6^{(2)}, \mathbf{a}_7^{(2)}, \mathbf{a}_4^{(2)}$, ***i.e.*** $(\Delta_1, \Delta_6, \Delta_7$ **and** $\Delta_4)$.

$\Delta_1 =$ (second row of $\mathbf{B}_2^{-1}$) $(\mathbf{a}_1^{(2)}) = (0, 1, 16/7, 3/7, 0)\ (-1, -4, 1, 2, 1) = -6/7;$

$\Delta_6 = (0, 1, 16/7, 3/7, 0)\ (0, 0, 1, 0, 0) = 16/7.\ \Delta_7 = (0, 1, 16/7, 3/7, 0)\ (0, 0, 0, 1, 0) = 3/7;$

$\Delta_4 = (0, 1, 16/7, 3/7, 0)\ (1, -1, 0, 0, 1) = -1.$

Since Δ_1 and Δ_4 are still negative, further improvement of x_5 is possible.

Step 2. ***Determination of the entering vector*** $\mathbf{a}_k^{(2)}$.

The entering vector $\mathbf{a}_k^{(2)}$ corresponds to such value of k which is obtained by the cirterion :

$$\Delta_k = \min\,[\Delta_1, \Delta_6, \Delta_7, \Delta_4] = \min\left[-\frac{6}{7}, \frac{16}{7}, \frac{3}{7}, -1\right] = -1 = \Delta_4.\ \text{Hence } k = 4.$$

So introduce $\mathbf{a}_4^{(2)}$.

Step 3. ***Determination of the leaving vector*** $\beta_r^{(2)}$, ***given the entering vector*** $\mathbf{a}_4^{(2)}$.

Compute

$$\mathbf{X}_4^{(2)} = \mathbf{B}_2^{-1}\,\mathbf{a}_4^{(2)} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 16/7 & 3/7 & 0 \\ 0 & 0 & 5/7 & -3/7 & 0 \\ 0 & 0 & -1/7 & 2/7 & 0 \\ 0 & 0 & -9/7 & 4/7 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

After completing the column $\mathbf{X}_k^{(2)} = \mathbf{X}_4^{(2)}$ in *Table 6·19*, add one more column for minimum ratio rule. Minimum ratio rule indicates that $\boxed{1}$ is the key element. Hence remove the vector $\beta_3^{(2)}$ from the basis.

Step 4. ***Determination of next improved value of*** x_5.

Introduce $\mathbf{a}_4^{(2)}$ and remove $\beta_3^{(2)}$, to obtain the following transformation table.

Table 6·20

Variables in the basis	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	$\mathbf{X}_B^{(2)}$	$\mathbf{X}_k^{(2)}$ $(k=1)$	Min. Ratio $(\mathbf{X}_B/\mathbf{X}_K)$ $(k=1)$
z	1	0	16/7	−4/7	−1	90/7	−6/7	
x_5	0	1	1	1	1	$x_5 = 0$ (note)	0	
x_2	0	0	5/7	−3/7	0	15/7	−1/7	—
x_3	0	0	−1/7	2/7	0	25/7	3/7	25/3
x_4	0	0	−9/7	4/7	1	15/7	$\boxed{6/7}$	15/6 ←
					↓		↑	

Additional Table (6·20)′

$\mathbf{a}_1^{(2)}$	$\mathbf{a}_6^{(2)}$	$\mathbf{a}_7^{(2)}$	$\mathbf{a}_8^{(2)}$
−1	0	0	0
−4	0	0	0
1	1	0	0
2	0	1	0
1	0	0	1
√	×	×	×

Since the maximum value of x_5 becomes zero, consequently other artificial variables x_6, x_7, x_8 also become zero. Hence phase I ends at this stage. Cross (×) the artificial vectors $\mathbf{a}_6^{(2)}, \mathbf{a}_7^{(2)}, \mathbf{a}_8^{(2)}$ from the additional *Table (6·20)′* as the process of *Phase I* is now complete.

Now enter *Phase II* to maximize z (not x_5).

Phase II. First, test whether the value $z = 90/7$ [as obtained in Table (6·20)] is maximum.

Step 1. ***Computation of*** Δ_j **for** $\mathbf{a}_1^{(2)}$ ***only (i.e.*** Δ_1***).***

$$\Delta_1 = (\text{first row of } \mathbf{B}_2^{-1})\,(\mathbf{a}_1^{(2)}) = (1, 0, 16/7, -4/7, -1)\,(-1, -4, 1, 2, 1) = -\frac{6}{7} \qquad \textbf{(Note)}$$

There is no need to compute $\Delta_6, \Delta_7, \Delta_8$; because the corresponding artificial vectors $\mathbf{a}_6^{(2)}, \mathbf{a}_7^{(2)}$, and $\mathbf{a}_8^{(2)}$ have been ignored from the additional Table (6·20)′.

Since Δ_1 is negative, *Table 6·20* does not give optimum solution. Hence proceed to improve the solution :

$$z = 90/7,\ x_2 = 15/7,\ x_3 = 25/7,\ x_4 = 15/7.$$

Step 2. ***Determination of the entering vector*** $\mathbf{a}_k^{(2)}$.

The entering vector $\mathbf{a}_k^{(2)}$ corresponds to such value of k which can be obtained by the criterion :

$$\Delta_k = \min.\ (\Delta_1) = \Delta_1. \quad \text{Hence} \quad k = 1.$$

So we must enter $\mathbf{a}_1^{(2)}$.

Step 3. Determination of the leaving vector, given the entering vector $\mathbf{a}_1^{(2)}$.

Compute
$$\mathbf{X}_1^{(2)} = \mathbf{B}_2^{-1}\,\mathbf{a}_1^{(2)} = \begin{bmatrix} 1 & 0 & 16/7 & -4/7 & -1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 5/7 & -3/7 & 0 \\ 0 & 0 & -1/7 & 2/7 & 0 \\ 0 & 0 & -9/7 & 4/7 & 1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6/7 \\ 0 \\ -1/7 \\ 3/7 \\ 6/7 \end{bmatrix}$$

After completing the column $\mathbf{X}_k^{(2)} = \mathbf{X}_1^{(2)}$ in *Table 6·20*, add one more column for minimum ratio rule, which indicates that $\boxed{6/7}$ is the key element. Hence remove $\beta_3^{(2)}$ from the basis.

Step 4. ***Determination of the improved solution.***

Applying usual rules of matrix transformation to obtain the following table of improved solution.

Table 6·21

Variables in the basis	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	$\mathbf{X}_B^{(2)}$	$\mathbf{X}_k^{(2)}$
z	1	0	1	0	0	15	
x_5	1	1	1	1	1	0	
x_2	0	0	3/6	−2/6	1/6	5/2	
x_3	0	0	3/6	0	−3/6	5/2	
x_1	0	0	−9/6	−4/6	7/6	5/2	

$\mathbf{a}_4^{(2)}$
0
0
1
0
0

Test whether the solution $x_1 = x_2 = x_3 = 5/2$, $z = 15$ is optimum or not. For this, compute

$$\Delta_4 = (\text{first row of } \mathbf{B}_2^{-1})\ (\mathbf{a}_4^{(2)}) = (1, 0, 1, 0, 0)\ (0, 0, 1, 0, 0) = 1 \text{ (which is positive)}$$

Hence the optimum solution is obtained as : $x_1 = x_2 = x_3 = 5/2$, max. $z = 15$.

6.12. MORE EXAMPLES ON STANDARD FORM – II

Example 6. *Solve the following problem by revised simplex method :*

Min. $z = x_1 + 2x_2$ *subject to* $2x_1 + 5x_2 \geq 6,\ x_1 + x_2 \geq 2$, and $x_1, x_2 \geq 0$ **[Meerut (TDC) 90]**

Solution. Converting the objective function of minimization into maximization, we get

$$\text{Max. } z' = -x_1 - 2x_2 \text{, where } z' = -z .$$

Now, writing the given problem in the proper form

$$\begin{array}{llll} z' + x_1 + 2x_2 & & = 0 & \\ -3x_1 - 6x_2 + x_3 + x_4 + x_5 & & = -8 & \text{where } x_5 = -(x_6 + x_7) \\ 2x_1 + 5x_2 - x_3 & + x_6 & = 6 & \\ x_1 + x_2 \quad - x_4 & \quad + x_7 & = 2 . & \end{array}$$

Next, express this system of constraint equations in matrix form as follows :

$$\begin{array}{cccccccc} \mathbf{a}_0^{(2)} & \mathbf{a}_1^{(2)} & \mathbf{a}_2^{(2)} & \mathbf{a}_3^{(2)} & \mathbf{a}_4^{(2)} & \mathbf{a}_5^{(2)} & \mathbf{a}_6^{(2)} & \mathbf{a}_7^{(2)} \\ (\mathbf{e}_1) & & & & & (\mathbf{e}_2) & (\beta_1^{(2)}) & (\beta_2^{(2)}) \end{array}$$

$$\begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z' \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ 6 \\ 2 \end{bmatrix} .$$

Now enter Phase I.

Phase I. In this phase, maximize x_5 (not z') and form the following table for first iteration.

Table 6·22

Variables in the basis	$\mathbf{B}_2^{-1}$: e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\mathbf{X}_B^{(2)}$	$\mathbf{X}_k^{(2)}$ $(k=2)$	Min. Ratio $(\mathbf{X}_B/\mathbf{X}_2)$
z	1	0	0	0	0	2	
x_5	0	1	0	0	−8	−6	
$\leftarrow x_6$	0	0	1	0	6	$\boxed{5}$	6/5 ←
x_7	0	0	0	1	2	1	2/1
			↓			↑	

Additional Table

$\mathbf{a}_1^{(2)}$	$\mathbf{a}_2^{(2)}$	$\mathbf{a}_3^{(2)}$	$\mathbf{a}_4^{(2)}$
1	2	0	0
−3	−6	1	1
2	5	−1	0
1	1	0	−1

First Iteration

Step 1. ***Computation of*** Δ_j ***for*** $\mathbf{a}_1^{(2)}, \mathbf{a}_2^{(2)}, \mathbf{a}_3^{(2)}, \mathbf{a}_4^{(4)}$, ***i.e.,*** $\Delta_1, \Delta_2, \Delta_3$ ***and*** Δ_4.

$\Delta_1 = (\text{second row of } \mathbf{B}_2^{-1})\ (\mathbf{a}_1^{(2)}) = (0, 1, 0, 0)\ (1, -3, 2, 1) = -3$, $\Delta_2 = (0, 1, 0, 0)\ (2, -6, 5, 1) = -6$

$\Delta_3 = (0, 1, 0, 0)\ (0, 1, -1, 0) = 1$, $\Delta_4 = (0, 1, 0, 0)\ (0, 1, 0, -1) = 1$.

Step 2. ***Determination of the entering vector*** $\mathbf{a}_k^{(2)}$.

The entering vector $\mathbf{a}_k^{(2)}$ corresponds to such value of k which is obtained by the criterion

$$\Delta_k = \min. [\Delta_1, \Delta_2, \Delta_3, \Delta_4], = \min. [-3, -6, 1, 1] = -6 = \Delta_2 . \text{ Hence } k = 2 .$$

So the entering vector is determined to be $\mathbf{a}_2^{(2)}$, x_2 will be the entering variable.

Step 3. ***Determination of the leaving vector*** $\beta_r^{(2)}$ ***, given the entering vector*** $\mathbf{a}_2^{(2)}$.

First, compute $\mathbf{X}_2^{(2)}$ corresponding to the entering vector $\mathbf{a}_2^{(2)}$.

$$\mathbf{X}_2^{(2)} = \mathbf{B}_2^{-1}\,\mathbf{a}_2^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ 5 \\ 1 \end{bmatrix} = \begin{pmatrix} 2 \\ -6 \\ 5 \\ 1 \end{pmatrix}.$$

Now, it is possible to complete the column $\mathbf{X}_k^{(2)} = \mathbf{X}_2^{(2)}$ of Table 6·22.

Apply the min ratio rule within *Table 6·22*. This rule shows that $\boxed{5}$ is the '*key element*' corresponding to which $\beta_1^{(2)}$ $[\mathbf{a}_6^{(2)}]$ must be removed. Hence x_6 will become the departing variable (x_6 will have the value zero at the next iteration).

Step 4. ***Determination of first improved value of x_5.***

In the following first intermediate coefficient matrix, apply the usual rules of transformation, to obtain the second matrix

$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\mathbf{X}_B^{(2)}$	$\mathbf{X}_2^{(2)}$
0	0	0	2
0	0	-8	-6
1	0	6	$\boxed{5}$
0	1	2	1

− 2/5	0	− 12/5	0
6/5	0	− 4/5	0
1/5	0	6/5	1
− 1/5	1	4/5	0

and construct *Table* 6·23 for second iteration.

Table 6·23

Variables in the basis	$\mathbf{B}_2^{-1}$					$\mathbf{X}_k^{(2)}$ ($k = 1$)	Min. Ratio ($\mathbf{X}_B/\mathbf{X}_1$)
	$\mathbf{e}_1$	$\mathbf{e}_2$	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\mathbf{X}_B^{(2)}$		
z	1	0	− 2/5	0	− 12/5	1/5	
x_5	0	1	6/5	0	− 4/5	− 3/5	
$\rightarrow x_2$	0	0	1/5	0	6/5	2/5	$\frac{6}{5}/\frac{2}{5} = 3$
$\leftarrow x_7$	0	0	− 1/5	1 ↓	4/5	3/5 ↑	$\frac{4}{5}/\frac{3}{5} = \frac{4}{3} \leftarrow$

Additional Table

$\mathbf{a}_1^{(2)}$	$\mathbf{a}_6^{(2)}$	$\mathbf{a}_3^{(2)}$	$\mathbf{a}_4^{(2)}$
1	0	0	0
−3	0	1	1
2	1	−1	0
1	0	0	−1
↑			

Here $x_5 = -4/5$. Now enter the second iteration.

Second Iteration

Step 1. ***Computations of Δ_j for $\mathbf{a}_1^{(2)}, \mathbf{a}_6^{(2)}, \mathbf{a}_3^{(2)}, \mathbf{a}_4^{(2)}$, i.e. $(\Delta_1, \Delta_6, \Delta_3, \Delta_4)$.***

$\Delta_1 = (0, 1, 6/5, 0)\,(1, -3, 2, 1) = -3/5,$ $\qquad \Delta_6 = (0, 1, 6/5, 0)\,(0, 0, 1, 0) = 6/5$

$\Delta_3 = (0, 1, 6/5, 0)\,(0, 1, -1, 0) = -1/5,$ $\qquad \Delta_4 = (0, 1, 6/5, 0)\,(0, 1, 0, -1) = 1$.

Step 2. ***Determination of the entering vector $\mathbf{a}_k^{(2)}$.***

To find k, we have

$\Delta_k = \min.\ [\Delta_1, \Delta_6, \Delta_3, \Delta_4] = \min.\ [-3/5, 6/5, -1/5, 1] = -3/5 = \Delta_1$. Hence $k = 1$.

Now enter the vector $\mathbf{a}_k^{(2)} = \mathbf{a}_1^{(2)}$. Hence, x_1 will be the entering variable.

Step 3. ***Determination of the leaving vector $\beta_r^{(2)}$, given the entering vector $\mathbf{a}_1^{(2)}$***

Compute $\mathbf{X}_1^{(2)}$ corresponding to vector $\mathbf{a}_1^{(2)}$.

$$\mathbf{X}_1^{(2)} = \mathbf{B}_2^{-1}\,\mathbf{a}_1^{(2)} = \begin{bmatrix} 1 & 0 & -2/5 & 0 \\ 0 & 1 & 6/5 & 0 \\ 0 & 0 & 1/5 & 0 \\ 0 & 0 & -1/5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 \\ -3/5 \\ 2/5 \\ 3/5 \end{bmatrix}$$

Compute the column $\mathbf{X}_k^{(2)} = \mathbf{X}_1^{(2)}$ and apply the rule of min. $(\mathbf{X}_B/\mathbf{X}_1)$ in the last column of *Table 6·23* . It is found that 3/5 is the 'key element' which indicates that $\beta_2^{(2)}$ $[\mathbf{a}_7^{(2)}]$ should be removed. Hence x_7 will be the departing variable.

Step 4. ***Determination of the second improved value of*** x_5.

The intermediate coefficient matrix is given below. Applying the usual rules of matrix transformation, get the second matrix.

$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\mathbf{X}_B^{(2)}$	$\mathbf{X}_k^{(2)}$
$-2/5$	0	$-12/5$	$1/5$
$6/5$	0	$-4/5$	$-3/5$
$1/5$	0	$6/5$	$2/5$
$-1/5$	1	$4/5$	$\boxed{3/5}$

$-1/3$	$-1/3$	$-8/3$	0
1	1	0	0
$1/3$	$-2/3$	$2/3$	0
$-1/3$	$5/3$	$4/3$	1

Now table for improved value of x_5 becomes :

Table 6·24

Variables in the basis	$\mathbf{e}_1$	$\mathbf{e}_2$	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\mathbf{X}_B^{(2)}$	$\mathbf{X}_k^{(2)}$
z'	1	0	$-1/3$	$-1/3$	$-8/3$	
x_5	0	1	1	1	0	
x_2	0	0	$1/3$	$-2/3$	$2/3$	
$\rightarrow x_1$	0	0	$-1/3$	$5/3$	$4/3$	

Addl. Table

$\mathbf{a}_7^{(2)}$	$\mathbf{a}_6^{(2)}$	$\mathbf{a}_3^{(2)}$	$\mathbf{a}_4^{(2)}$
0	0	0	0
0	0	1	1
0	1	-1	0
1	0	0	-1
×	×		

Since max. $x_5 = 0$, we enter Phase II to maximize z' instead of x_5. Artificial column vectors $\mathbf{a}_7^{(2)}$ and $\mathbf{a}_6^{(2)}$ in the additional table, may be overlooked.

Phase II.

Step 1. ***Computations of*** Δ_j **for** $\mathbf{a}_3^{(2)}$ **and** $\mathbf{a}_4^{(2)}$, ***i.e.*** Δ_3, Δ_4

$\Delta_3 =$ (first row of $\mathbf{B}_2^{-1}$) $(\mathbf{a}_3^{(2)}) = (1, 0, -1/3, -1/3)\ (0, 1, -1, 0) = 1/3$.

$\Delta_4 =$ (first row of $\mathbf{B}_2^{-1}$) $(\mathbf{a}_4^{(2)}) = (1, 0, -1/3, -1/3)\ (0, 1, 0, -1) = 1/3$.

Since Δ_3 and Δ_4 are positive, the solution

$x_1 = 4/3$, $x_2 = 2/3$, $z' = -8/3$ $(z = 8/3)$, is optimal.

Note. Do not compute Δ_7, Δ_6 here, because there is no need to consider the corresponding artificial vectors $\mathbf{a}_7^{(2)}$ and $\mathbf{a}_6^{(2)}$ for Phase II.

Example 7. *Solve the following problem by revised simplex method.*

Min. $z = 2x_1 + x_2$, *subject to* $3x_1 + x_2 \le 3, 4x_1 + 3x_2 \ge 6, x_1 + 2x_2 \le 3, x_1, x_2 \ge 0$.

Solution. Converting the objective function from minimization to maximization,

$$\text{max. } z' = -2x_1 - x_2, \text{ where } z' = -z.$$

The system of constraint equations suitable for standard form II will become

$$\begin{array}{llllllll} z' + 2x_1 + x_2 & & & & & & = & 0 \\ -8x_1 - 6x_2 + x_3 - x_4 + x_5 & & & & & & = & -12 \\ 3x_1 + x_2 & & & & + x_6 & & = & 3 \\ 4x_1 + 3x_2 - x_3 & & & & & + x_7 & = & 6 \\ x_1 + 2x_2 & + x_4 & & & & + x_8 & = & 3. \end{array}$$

Note. The artificial variable x_8 is introduced together with the slack variable x_4 in the last equation to obtain the basis matrix $\mathbf{B}_2$ as identity matrix.

Expressing the system of constraint equations in matrix form :

$$\begin{array}{ccccccccc} \mathbf{a}_0^{(2)} & \mathbf{a}_1^{(2)} & \mathbf{a}_2^{(2)} & \mathbf{a}_3^{(2)} & \mathbf{a}_4^{(2)} & \mathbf{a}_5^{(2)} & \mathbf{a}_6^{(2)} & \mathbf{a}_7^{(2)} & \mathbf{a}_8^{(2)} \\ (\mathbf{e}_1) & & & & & (\mathbf{e}_2) & \beta_1^{(2)} & \beta_2^{(2)} & \beta_3^{(2)} \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8 & -6 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 3 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z' \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ -12 \\ 3 \\ 6 \\ 2 \end{bmatrix}$$

and enter Phase I.

Phase I. In this phase, maximize x_5 (not z'). Construct *Table 6·25* for first iteration.

Table 6·25

Variables in the basis	B_2^{-1}							Min
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	$X_B^{(2)}$	$X_k^{(2)} (k=1)$	$\left(\frac{X_B}{X_1}\right)$
z'	1	0	0	0	0	0	2	
x_5	0	1	0	0	0	–12	–8	
$\leftarrow x_6$	0	0	1	0	0	3	$\boxed{3}$	3/3 ←
x_7	0	0	0	1	0	6	4	6/4
x_8	0	0	0	0	1	3	1	3/1
			↓				↑	

Additional Table

$a_1^{(2)}$	$a_2^{(2)}$	$a_3^{(2)}$	$a_4^{(2)}$
2	1	0	0
–8	–6	1	–1
3	1	0	0
4	3	–1	0
1	2	0	1

First Iteration

Step 1. ***Computation of Δ_j for $a_1^{(2)}, a_2^{(2)}, a_3^{(2)}, a_4^{(2)}$, i.e. $\Delta_1, \Delta_2, \Delta_3, \Delta_4$.***

Δ_1 = (second row of B_2^{-1}) $(a_1^{(2)}) = (0, 1, 0, 0, 0)\,(2, -8, 3, 4, 1) = -8$;

$\Delta_2 = (0, 1, 0, 0, 0)\,(1, -6, 1, 3, 2) = -6$;

$\Delta_3 = (0, 1, 0, 0, 0)\,(0, 1, 0, -1, 0) = 1$; $\Delta_4 = (0, 1, 0, 0, 0)\,(0, -1, 0, 0, 1) = -1$.

Step 2. ***Determination of the entering vector $a_k^{(2)}$.***

The entering vector $a_k^{(2)}$ corresponds to such value of k which is obtained by the criterion

$$\Delta_k = \text{min.}\,[\Delta_1, \Delta_2, \Delta_3, \Delta_4], = \text{min.}\,[-8, -6, 1, -1] = -8 = \Delta_1. \text{ Hence } k = 1.$$

So enter $a_1^{(2)}$, that is, x_1 will be the entering variable.

Step 3. ***Determination of the leaving vector $\beta_r^{(2)}$, given the entering vector $a_1^{(2)}$.***

First, compute $X_1^{(2)}$ corresponding to the entering vector $a_1^{(2)}$.

$$X_1^{(2)} = B_2^{-1}\,a_1^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -8 \\ 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

and complete the column $X_k^{(2)} = X_1^{(2)}$ of *Table 6·25*.

Apply the minimum ratio rule in Table 6·25. This rule immediately gives the key element 3 corresponding to which the vector $\beta_1^{(2)}$, that is, $a_6^{(2)}$ must be removed. Hence x_6 will become the departing variable.

Step 4. ***Determination of the first improved value of x_5.***

Consider the intermediate coefficient matrix as follows. Then apply the usual rules of matrix transformation to obtain the second matrix.

$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	$X_B^{(2)}$	$X_1^{(2)}$
0	0	0	0	2
0	0	0	–12	–8
1	0	0	3	$\boxed{3}$
0	1	0	6	4
0	0	1	3	1

– 2/3	0	0	–2	0
8/3	0	0	–4	0
1/3	0	0	1	1
– 4/3	1	0	2	0
– 1/3	0	1	2	0

Construct *Table* 6·26 for second iteration.

Table 6·26

Varaibles in the basis	B_2^{-1}					$X_B^{(2)}$	$X_k^{(2)}$ $(k=2)$	Min. Ratio Rule :
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$			
z	1	0	– 2/3	0	0	–2	1/3	Min
x_5	0	1	8/3	0	0	–4	– 10/3	(X_B/X_2)
$\rightarrow x_1$	0	0	1/3	0	0	1	1/3	$1/\frac{1}{3}$
$\leftarrow x_7$	0	0	– 4/3	1	0	2	5/3	$2/\frac{5}{3}*$
x_8	0	0	– 1/3	0	1	2	5/3	$2/\frac{5}{3}*$

Additional Table

$a_6^{(2)}$	$a_2^{(2)}$	$a_3^{(3)}$	$a_4^{(2)}$
0	1	0	0
0	–6	1	–1
1	1	0	0
0	3.	–1	0
0	2	0	1
×	×		

Here $x_5 = -4$. Therefore, enter the second iteration.

Second Iteration

Step 1. ***Computation of*** Δ_j **for** $\mathbf{a}_6^{(2)}, \mathbf{a}_2^{(2)}, \mathbf{a}_3^{(2)}, \mathbf{a}_4^{(2)}$ ***i.e.*** $\Delta_6, \Delta_2, \Delta_3, \Delta_4$.

$\Delta_6 = (\text{second row of } \mathbf{B}_2^{-1})\,(\mathbf{a}_6^{(2)}) = (0, 1, 8/3, 0, 0)\,(0, 0, 1, 0, 0) = 8/3$

$\Delta_2 = (0, 1, 8/3, 0, 0)\,(1, -6, 1, 3, 2) = -10/3, \qquad \Delta_3 = (0, 1, 8/3, 0, 0)\,(0, 1, 0, -1, 0) = 1$

$\Delta_4 = (0, 1, 8/3, 0, 0)\,(0, -1, 0, 0, 1) = -1.$

Step 2. ***Determination of the entering vector*** $\mathbf{a}_k^{(2)}$.

The value of k to obtain $\mathbf{a}_k^{(2)}$ is determined by the criterion

$\Delta_k = \min.\,[\Delta_6, \Delta_2, \Delta_3, \Delta_4] = \min.\,[8/3, -10/3, 1, -1] = -10/3 = \Delta_2$. Hence $k = 2$. So introduce $\mathbf{a}_2^{(2)}$.

Step 3. ***Determination of the leaving vector*** $\beta_r^{(2)}$ ***, given the entering vector*** $\mathbf{a}_2^{(2)}$.

First, compute $\mathbf{X}_2^{(2)}$ corresponding to the entering vector $\mathbf{a}_2^{(2)}$. Therefore,

$$\mathbf{X}_2^{(2)} = \mathbf{B}_2^{-1}\,\mathbf{a}_2^{(2)} = \begin{bmatrix} 1 & 0 & -2/3 & 0 & 0 \\ 0 & 1 & 8/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & -4/3 & 1 & 0 \\ 0 & 0 & -1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -6 \\ 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -10/3 \\ 1/3 \\ 5/3 \\ 5/3 \end{bmatrix}$$

Now complete the column $\mathbf{X}_k^{(2)} = \mathbf{X}_2^{(2)}$ of *Table 6·26.*

Apply minimum ratio rule in the last column of Table 6·26 . It is found that the minimum ratio of (x_{Bi}/x_{i2}) is not unique but occurs for $i = 2, 3$. So we face the problem of degeneracy at this stage. Here it is necessary to resolve degeneracy in order to prevent cycling. Therefore, to find the unique minimum ratio apply the *Charne's Perturbation Technique* as discussed in ***Chapter 3.***

From *Table 6·26* , compute

$$\min_{i=2,3}\left[\frac{x_{i1}}{x_{i2}}\right] = \min\left[\frac{x_{21}}{x_{22}}, \frac{x_{31}}{x_{32}}\right] = \min\left[\frac{-4/3}{5/3}, \frac{-1/3}{5/3}\right] = \min\,[-\tfrac{4}{5}, -\tfrac{1}{5}] = -\tfrac{4}{5}.$$

Since the minimum is unique and attained at $i = 2$, the vector to be removed will be $\beta_2^{(2)}$.

Step 4. ***Determination of the second improved value of*** x_5 .

Consider the intermediate coefficient matrix as given below. Apply usual rules of matrix transformation to get the second matrix.

$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	$\mathbf{X}_B^{(2)}$	$\mathbf{X}_2^{(2)}$
−2/3	0	0	−2	1/3
8/3	0	0	−4	−10/3
1/3	0	0	1	1/3
−4/3	1	0	2	$\boxed{5/3}$
−1/3	0	1	2	5/3

−2/5	−1/5	0	−12/5	0
0	2	0	0	0
3/5	−1/5	0	3/5	0
−4/5	3/5	0	6/5	1
1	−1	1	0	0

Construct *Table 6·27* for next iteration.

Table 6·27

Variables in the basis	e_1	e_2	$\beta_1^{(1)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	$\mathbf{X}_B^{(2)}$	$\mathbf{X}_k^{(2)}$
z'	1	0	−2/5	−1/5	0	−12/5	
x_5	0	1	0	2	0	0	
x_1	0	0	3/5	−1/5	0	3/5	
$\to x_2$	0	0	−4/5	3/5	0	6/5	
x_8	0	0	1	−1	1	0	

Additional Table

$\mathbf{a}_6^{(2)}$	$\mathbf{a}_7^{(2)}$	$\mathbf{a}_3^{(2)}$	$\mathbf{a}_4^{(2)}$
0	0	0	0
0	0	1	−1
1	0	0	0
0	1	−1	0
0	0	0	1
×	×		

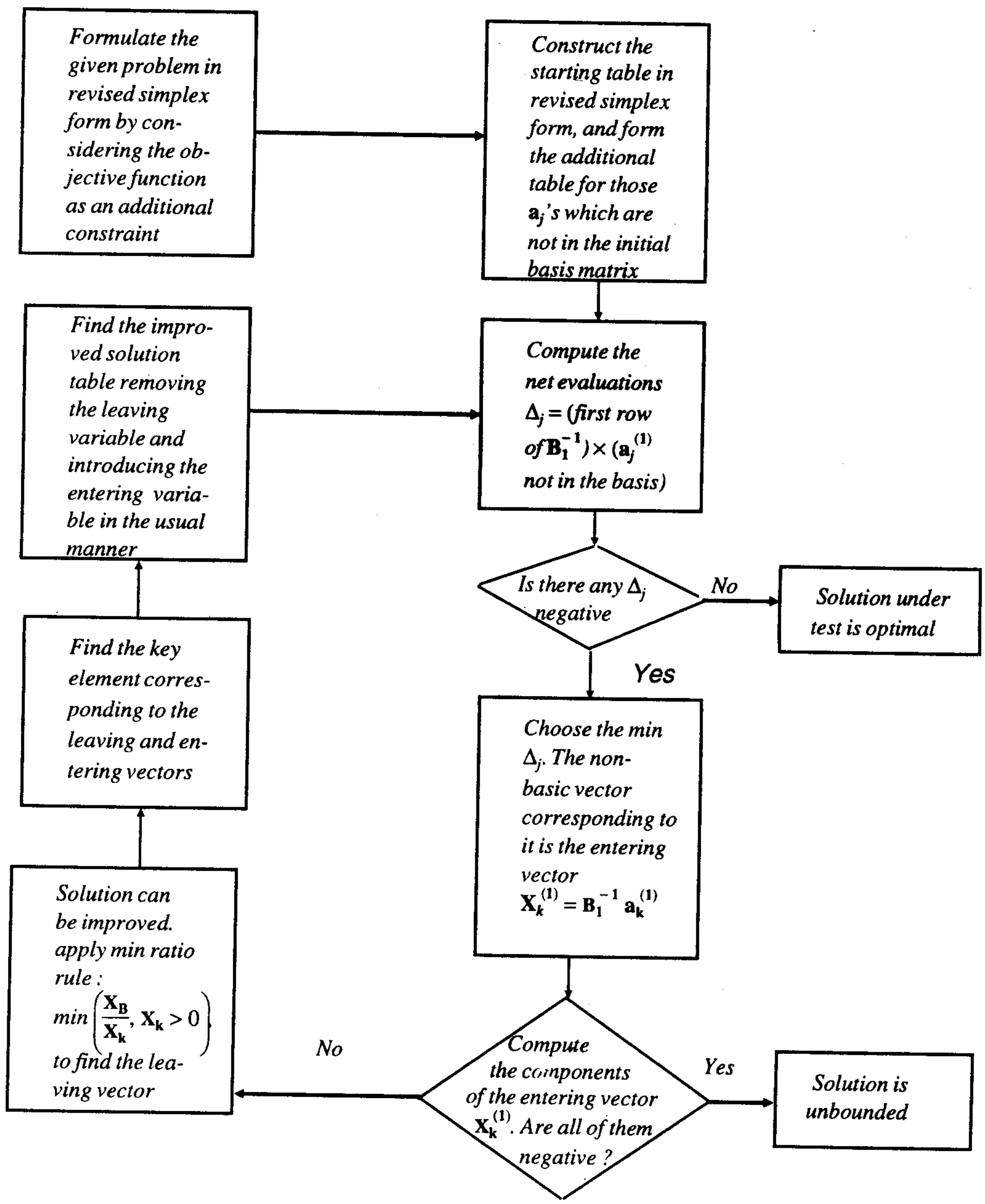
Here maximum x_5 comes out to be zero. Hence enter Phase II to maximize z'. Also cross out the artificial vectors $\mathbf{a}_6^{(2)}$ and $\mathbf{a}_7^{(2)}$ from the additional table at this stage.

Phase II. Step 1. ***Computations Δ_j for $\mathbf{a}_3^{(2)}$ and $\mathbf{a}_4^{(2)}$.***

$$\Delta_3 = (\text{first row of } \mathbf{B}_2^{-1})\,(\mathbf{a}_3^{(2)}) = (1, 0, -2/5, -1/5, 0)\,(0, 1, 0, -1, 0) = 1/5;$$
$$\Delta_4 = (1, 0, -2/5, -1/5, 0)\,(0, -1, 0, 0, 1) = 0\,.$$

Since Δ_3, and Δ_4 are ≥ 0, the solution is : $x_1 = 3/5$, $x_2 = 6/5$, $z' = -12/5$ (hence $z = 12/5$) is optimal.

Flowchart of Revised Simplex Method : Standard Form–I



Further, Δ_4 being zero shows that the problem has alternative optimal solutions.

EXAMINATION PROBLEMS

1. Min. $z = x_1 + x_2$, subject to
$x_1 + 2x_2 \geq 7$
$4x_1 + x_2 \geq 6$
$x_1, x_2 \geq 0$.
[Delhi, B.Sc. (Math.) 89, 87, 77; Raj. 80]
[Ans. $x_1 = 5/7$, $x_2 = 22/7$, min. $z = 27/7$]

2. Using artificial variables, solve by revised simplex method :
Max. $z = -2x_1 - 4x_2 - x_3$, subject to
$x_1 + 2x_2 - x_3 \leq 5, 2x_1 - x_2 + 2x_3 = 2, -x_1 + 2x_2 + 2x_3 \geq 1$
$x_1, x_2, x_3 \geq 0$.
[Ans. $x_1 = 1/2$, $x_2 = 0. < x_3 = 2/3$, $z^* = -4/3$.
Alternative solutions also exist].

6.13. ADVANTAGES AND DISADVANTAGES

Advantages :

1. The method automatically generates the inverse of the current basis matrix and the new basic feasible solution as well.
2. It provides more information at lesser computational effort.
3. It requires lesser computations than the ordinary simplex method.
4. A less number of entries are needed in each table of revised simplex method.
5. The control of rounding-off errors occurs when a digitial computer is used.

Disadvantages :

In solving the numerical problems side computations are also required, therefore more computational mistakes may occur in comparison to original simplex method.

Note. Students are advised to prepare a flow-chart for standard form–II.

Q. 1. Compare the revised simplex method with simplex method and bring out the salient points of differences.
2. What are the advantages and disadvantages of revised simplex method over the original simplex method.
3. When a revised simplex method is advantageous ?

SELF-EXAMINATION PROBLEMS

1. Formulate a linear programming problem in the form of revised simplex.
2. Develop the computational algorithm for solving a linear programming problem by revised simplex method.

Use revised simplex method to solve :

3. Max. $z = 3x_1 + 5x_2$
subject to the constraints :
$x_1 \leq 4$
$x_2 \leq 6$
$3x_1 + 2x_2 \leq 18$
$x_1, x_2 \geq 0$.
[Ans. $x_1 = 2$, $x_2 = 6$, max. $z = 36$]

4. Max. $z = x_1 + x_2 + 3x_3$
subject to the constraints :
$3x_1 + 2x_2 + x_3 \leq 3$
$2x_1 + x_2 + 2x_2 \leq 2$
$x_1, x_2, x_3 \geq 0$.
[Ans. $x_1 = x_2 = 0$, $x_2 = 1$, max. $z = 3$].

5. Max. $z = 5x_1 + 3x_2$,
subject to the constraints.
$4x_1 + 5x_2 \geq 10$
$5x_1 + 2x_2 \leq 10$
$3x_1 + 8x_2 \leq 12$
$x_1, x_2 \geq 0$.
[Ans. $x_1 = 28/17$, $x_2 = 15/17$, max. $z = 185/17$]

6. Min. $z = 3x_1 + x_2$
subject to the constraints :
$x_1 + x_2 \geq 1$
$2x_1 + x_2 \geq 0$
$x_1, x_2 \geq 0$.
[Ans. $x_1 = 0$, $x_2 = 1$, min. $z = 1$]

7. Min. $z = 4x_1 + 2x_2 + 3x_3$,
subject to the constraints :
$2x_1 + 4x_3 \geq 5$
$2x_1 + 4x_2 + x_3 \geq 4$
$x_1, x_2, x_3 \geq 0$
[Ans. $x_1 = 0$, $x_2 = 11/12$,
$x_3 = 5/4$, min. $z = 67/12$].

8. Max. $z = x_1 + 2x_2 + 3x_3 + 4x_4$,
subject to the constraints :
$3x_1 + 2x_2 + 3x_3 - x_4 \leq 25$
$-2x_1 + x_2 - 2x_3 - x_4 \geq 5$
$2x_1 + 2x_2 + x_3 + x_4 \geq 0$
$x_1, x_2, x_3, x_4 \geq 0$.
[Ans. $x_1 = x_2 = x_3 = 0$, $x_4 = 20$, max. $z = 80$].

9. Max. $z = 6x_1 - 2x_2 + 3x_3$, s.t. $2x_1 - x_2 + 2x_3 \leq 2$, $x_1 + 4x_3 \leq 4$, and $x_1, x_2, x_3 \geq 0$.
[Ans. $x_1 = 4$, $x_2 = 6$, $x_3 = 0$, $z = 12$]

10. If the factory receives an inquiry about the possibility of producing a fourth product *D* which calls for one hour each of cutting and assembleing time, and 2 hours of grinding time per unit and would contribute a profit of Rs. 3 per unit, should the order be sought after ? If so, how much of the fourth product *D* should the factory promise to deliver ? Solve the problem using revised simplx method.

***

7

DUALITY IN LINEAR PROGRAMMING

7.1. INTRODUCTION : CONCEPT OF DUALITY

One of the most important discoveries in the early development of linear programming was the concept of duality and its division into important branches. The discovery disclosed the fact that every linear programming problem has associated with it another linear programming problem. The original problem is called the "*primal*" while the other is called its "*dual*". It is important to note that, in general, either problem can be considered the primal, with the remaining problem its dual. The relationship between the '*primal*' and '*dual*' problems is actually a very intimate and useful one. The optimal solution of either problem reveals information concerning the optimal solution of the other. If the optimal solution to one is known, then the optimal solution of the other is readily available. This fact is important because the situation can arise where the dual is easier to solve than the primal.

7·1–1. Concept of Duality in Linear Programming.

In order to make the concept of duality clear, we consider the following diet problem of our common interest.

The amounts of two vitamins v_1 and v_2 *per unit present in two different foods* F_1 and F_2 *respectively are given in the following table :*

Vitamin	Food		Minimum Daily Requirement (units)
	F_1	F_2	
v_1	5	7	80
v_2	6	11	100
Cost per unit	Rs. 10	Rs. 15	

The problem is to determine the minimum quantities of two foods F_1 and F_2 *so that the minimum daily requirement of two vitamins is met and that at the same time, the cost of purchasing these quantities of* F_1 and F_2 *is minimum.*

To formulate this problem mathematically, let x_1 and x_2 be the number of units of food F_1 and F_2 to be purchased respectively. The problem is to find the values of x_1 and x_2 so as :

$$\begin{aligned}
&\textbf{To minimize } z_x = 10x_1 + 15x_2\\
&\textbf{subject to the constraints :}\\
&\qquad 5x_1 + 7x_2 \geq 80\\
&\qquad 6x_1 + 11x_2 \geq 100\\
&\textbf{and} \qquad x_1, x_2 \geq 0
\end{aligned}$$

Here in the formulation of the problem, we have assumed that taking more than the minimum requirement is not harmful, and purchase of negative quantity is meaningless. This LPP will be considered as the *primal problem*.

Now associated with the above problem, we can consider a different problem.

Suppose there is a wholesale dealer selling two vitamins v_1 and v_2 *along with some other commodities. The retailers purchase the vitamins from him and from the two foods* F_1 and F_2 *(as given in above table). The dealer knows very well that the foods* F_1 and F_2 *have their market values only because of their vitamin contents. The problem of the dealer is to fix-up the maximum per unit selling prices for the two vitamins* v_1 and v_2 *in such a manner that the resulting prices of foods* F_1 and F_2 *do not exceed their existing market prices.*

To formulate this problem mathematically, let the dealer decide to fix-up two prices w_1 and w_2 per unit respectively. The dealer's problem is to determine the values of w_1 and w_2 so as :

To maximize $z_w = 80\,w_1 + 100\,w_2$
subject to the constraints :
$$5w_1 + 6w_2 \le 10$$
$$7w_1 + 11w_2 \le 15$$
and $w_1, w_2 \ge 0$.

This associated LPP is considered as the *dual* of the given primal.

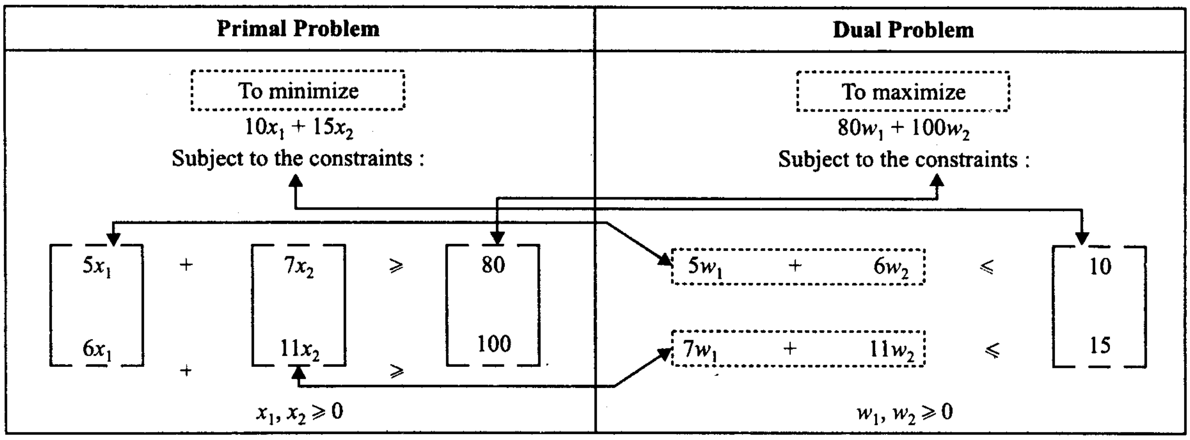
We absorve that both the above problems are symmetrical in the following sense :

(i) The costs associated with the objective function of one problem are just the requirements in the other's set of constraints.

(ii) The constraint coefficient matrix associated with one problem is simply the transpose of the constraint coefficient matrix associated with the other.

However, one of the problems is a maximization problem while the other is a *minimization* problem.

The obove primal dual construction relationship can be more easily understood by the following diagram :



Q. 1. Explain the concept of duality.

2. Discuss relationship between primal and its dual.

The concept of a dual problem formulation has often proved useful in science and engineering. Circuit theory, economics, and game theory are other examples of such cases. The dual linear programming problem has been, and continues to be, a powerful tool in the analysis of linear programming and related areas.

7.2. DEFINITION OF PRIMAL-DUAL PROBLEMS

7·2-1 Symmetric Primal-Dual Problems

Let us consider a linear programming problem in the following form, which may be called the *symmetric primal problem.*

Primal Problem : *Find* $x_1, x_2, x_3, \ldots, x_n$, *which maximize* $z_x = c_1x_1 + c_2x_2 + \ldots + c_nx_n$, *subject to*

$$\left.\begin{array}{rl} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1\,, \\ & a_{21}x_1 + a_{22}\,x_2 + \ldots + a_{2n}x_n \le b_2 \\ & \vdots \quad \vdots \qquad \vdots \quad \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \\ \text{and} & x_1\,, x_2\,, \ldots\,, x_n \ge 0 \end{array}\right. \qquad \ldots(7\cdot1)$$

where the sign of all parameters (a , b , c's) are arbitrary.

The dual of the above problem is obtained by,

(i) transposing the coefficient matrix;

(ii) interchanging the role of constant terms and the coefficients of the objective function;

(iii) reverting the inequalities;

(iv) minimizing the objective function instead of maximizing it.

Dual Problem : *Find* $w_1\,, w_2\,, w_3\,, \ldots\,, w_m$ *, which minimize* $z_w = b_1w_1 + b_2w_2 + \ldots + b_mw_m$ *, subject to*

$$\left.\begin{array}{rl} & a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m \ge c_1 \\ & a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m \ge c_2 \\ & \vdots \quad \vdots \qquad \vdots \quad \vdots \\ & a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}w_m \ge c_n\,, \\ \text{and} & w_1\,, w_2\,, \ldots\,, w_m \ge 0\,. \end{array}\right\} \qquad \ldots(7\cdot2)$$

Thus, by definition, (7·2) is the dual of (7·1), and $w_1\,, w_2\,, w_3\,, \ldots\,, w_m$ are called the *dual variables.*

The primal-dual relationship may be remembered more conveniently by using the following table :

$$\begin{array}{ccccc} & (x_1\,, \ldots\,, x_n) & & \text{Min.} \\ \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} & \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} & \le & \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \\ & \ge & & \\ \text{Max.} & (c_1\,, \ldots\,, c_n) & & \end{array}$$

Primal constraints should be read across the table while dual constraints should be read down the columns.

An example of a symmetric primal and its dual is given below :

Primal Problem : *Max.* $z_x = 3x_1 + 5x_2$ *, subject to* $x_1 \le 4$, $x_2 \le 6$, $3x_1 + 2x_2 \le 18$, *and* $x_1\,, x_2 \ge 0$.

The corresponding dual problem is the following :

Dual Problem : Min. $z_w = 4w_1 + 6w_2 + 18w_3$ *, subject to* $w_1 + 3w_3 \ge 3$, $w_2 + 2w_3 \ge 5$ and $w_1, w_2, w_3 \ge 0$.

7·2-2 Matrix Form of Symmetric Primal and Its Dual

Primal Problem. Find a column vector $\mathbf{X} \in R^n$, which maximizes $z_x = \mathbf{CX}$, $\mathbf{C} \in R^n$*(primal objective function)* subject to

$$\mathbf{AX} \le \mathbf{b},\ \mathbf{b} \in R^m\,, \mathbf{X} \ge \mathbf{0}\,. \qquad \ldots(7\cdot3)$$

where A is an $m \times n$ real matrix.

Dual Problem. Find a column vector $\mathbf{W} \in R^m$, which minimizes $z_w = \mathbf{b}^T\mathbf{W}$, $\mathbf{b} \in R^m$ *(dual objective function)* subject to

$$\mathbf{A}^T\mathbf{W} \ge \mathbf{C}^T,\ \mathbf{C} \in R^n,\ \ \mathbf{W} \ge \mathbf{0}\,, \qquad \ldots(7\cdot4)$$

where $\mathbf{W} = (w_1\,, w_2\,, \ldots\,, w_m)$ and $\mathbf{A}^T\,, \mathbf{b}^T\,, \mathbf{C}^T$ are the transpose of $\mathbf{A}\,, \mathbf{b}$, and $\mathbf{C}$ (given in the primal) respectively.

7·2-3 Unsymmetric Primal-Dual Problems

Primal Problem. Find a column vector $\mathbf{X} \in R^n$, which maximizes $z_x = \mathbf{CX},\ \mathbf{C} \in R^n$, subject to

$$\mathbf{AX} = \mathbf{b},\ \mathbf{X} \ge \mathbf{0},\ \mathbf{b} \in R^m$$

where $\mathbf{A}$ is an $m \times n$ real matrix.

Dual Problem. Find a column vector $\mathbf{W} \in R^m$, which minimizes $z_w = \mathbf{b}^T\mathbf{W}$, subject to $\mathbf{A}^T\,\mathbf{W} \ge \mathbf{C}^T$.

The final table of the primal problem indicates that the marginal value of raw material A is Rs. zero; for B is Rs. 10 per unit; and for C is Rs. 10 per unit. Thus if the manager sells the raw materials A, B and C at price Rs. 0, Rs. 10 and Rs. 10 per unit respectively, he will get the same contribution of Rs. 1050 which he is going to fetch in case he utilizes these resources for production of the three products X, Y and Z.

Example 14. *XYZ manufacturing company operates a three-shift system at one of its plants. In a certain section of the plant, the number of operators required on each of the three shifts is as follows :*

Shift	*Number of operators*
Day (6 a.m. to 2 p.m.)	50
Afternoon (2 p.m. to 10 p.m.)	24
Night (10 p.m. to 6 p.m.)	10

The company pays its operators at the basic rate of Rs. 10 per hour for those working on the day shift. For the afternoon and night shifts, the rates are one and a half times the basic rate and twice the basic rate respectively. In agreement with each operator at the commencement of his employment, he is allocated to one of three schemes A, B, or C. These are as follows :

(A) Work (on average) one night shift, one afternoon shift, and two day shifts in every four shifts.

(B) Work (on average) equal number of day and afternoon shifts.

(C) Work day shifts only.

In schemes A and B, it is necessary to work strictly alternating sequences of specified shifts, as long as the correct proportion of shifts as worked in the long run.

(i) Formulate a linear programming model to obtain the required number of operators at minimum cost.

(ii) By solving the dual of the problem, determine how many operators must be employed under each of the three schemes. Does this result in over-provision of operators on any one of the three shifts ?

[Bombay (M.M.S.,) Nov. 98]

Solution.

Primal problem :

$$\text{Minimize } Z = 20 \times \frac{1}{4}x_1 + 15\left(\frac{1}{4}x_1 + \frac{1}{2}x_2\right) + 10\left(\frac{1}{2}x_1 + \frac{1}{2}x_2 + x_3\right)$$

$$= \frac{55}{4}x_1 + \frac{25}{2}x_2 + 10x_3$$

subject to the constraints

$$\frac{1}{4}x_1 \geq 10$$

$$\frac{1}{4}x_1 + \frac{1}{2}x_2 \geq 24$$

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 + x_3 \geq 50$$

$$x_1, x_2, x_3 \geq 0$$

where x_1, x_2, x_3 are number of operators employed under scheme A, B and C respectively.

Dual problem :

$$\text{Max. } Z^* = 10y_1 + 24y_2 + 50y_3$$

subject to the constraints

$$\frac{1}{4}y_1 + \frac{1}{4}y_2 + \frac{1}{2}y_3 \leq \frac{55}{4}$$

$$\frac{1}{2}y_2 + \frac{1}{2}y_3 \leq \frac{25}{2}$$

$$y_3 \leq 10$$

$$y_1, y_2, y_3 \geq 0$$

where y_1, y_2, y_3 are shadow prices (or worth) per unit of resources– operators in three shifts respectively

The dual problem can now be solved by using simplex method after introducing slack variables s_1, s_2 and s_3 in the above constraints as follows :

$$\text{Max. } Z = 10y_1 + 24y_2 + 50y_3 + 0.s_1 + 0.s_2 + 0.s_3$$

subject to the constraints

$$\frac{1}{4}y_1+\frac{1}{4}y_2+\frac{1}{2}y_3+s_1 = \frac{55}{4}$$
$$\frac{1}{2}y_2+\frac{1}{2}y_3+s_2 = \frac{25}{2}$$
$$y_3+s_3 = 10;\ y_1, y_2, y_3, s_1, s_2, s_3 \geq 0$$

from the final simplex tableau, the optimum solution to the primal problem is read as :

$$x_1 = 40,\ x_2 = 28,\ x_3 = 16$$

and the minimum total cost $= \frac{55}{4}\times 40+\frac{25}{2}\times 28+10\times 16 =$ Rs. 1,070.

EXAMINATION PROBLEMS

Use principle of duality to solve the following LP problems :

1. Max. $z=3x_1+2x_2$, s.t. $2x_1+x_2 \leq 5$, $x_1+x_2 \leq 3$, $x_1, x_2 \geq 0$
   [**Ans.** $x_1=2$, $x_2=1$, max. $z=8$; $w_1=1$, $w_2=2$, min. $z_w=8$].
2. Max. $z=x_1+6x_2$, s.t. $x_1+x_2 \geq x_1+3x_2 \leq 3$; $x_1, x_2 \geq 0$.
   [**Ans.** $x_1=3/2$, $x_2=1/2$, max. $z=9/2$; $w_1=3/2$, $w_2=5/2$, min. $z_w=9/2$].
3. Min. $z=2x_1+2x_2$, s.t. $2x_1+4x_2 \geq 1$, $x_1+2x_2 \geq 1$, $2x_1+x_2 \geq 1$; $x_1, x_2 \geq 0$.
   [**Ans.** $x_1=1/3$, $x_2=1/3$, max. $z=4/3$].
4. Max. $z=3x_1+4x_2$, s.t. $x_1-x_2 \leq 1$, $x_1+x_2 \geq 4$, $x_1, 3x_2 \leq 3$, $x_1 < x_2 \geq 0$.
   [**Ans.** Since the dual problem does not possess any optimum basic feasible solution, hence there exists an unbounded solution to the primal problem].
5. Max. $z=2x_1+x_2$, s.t. $x_1+2x_2 \leq 10$, $x_1+x_2 \leq 6$, $x_1-x_2 \leq 2$, $x_1-2x_2 \leq 1$; $x_1, x_2 \geq 0$.
   [**Ans.** $x_1=4$, $x_2=2$, max. $z=10$].
6. (a)Use simplex method to maximize $z=5x-2y+3z$, subject to the restrictions :
   $2x+2y-z \geq 2$, $3x-4y \leq 3$, $y+3z \leq 5$, where x, y, z, are non-negative variables.
   (b) Verify your solution using the dual of the problem given above.
   [**Ans.** $x=23/3$, $y=5$, $z=0$, max. $z=85/3$].
7. Apply simplex method to solve the following :
   Max. $z=30x_1+23x_2+29x_3$, subject to, $6x_1+5x_2+3x_3 \leq 26$; $4x_1+2x_2+5x_3 \leq 7$, every $x_j \geq 0$.
   Also read the solution to the dual of the above problem from the final tableau.
   [**Ans.** $x_1=0$, $x_2=7/2$, $x_3=0$, max. $z=16/2$; for dual problem : $w_1=0$, $w_2=23/2$; min. $z_w=161/2$].
8. Apply the principle of duality to solve the LP problem :
   Max. $z=3x_1+2x_2$, subject to the constraints : $x_1+x_2 \geq 1$, $x_1+x_2 \leq 7$, $x_1+2x_2 \leq 10$, $x_2 \leq 3$; $x_1, x_2, \geq 0$.
   [**Ans.** For dual : $w_1=0$, $w_2=3$, $w_3=0$, $w_4=0$ and min. $z=21$; For primal : $x_1=7$, $x_2=0$, max. $z=21$].
9. Solve the dual of the following problem by the simplex method :
   Max. $z=2x_1+3x_2+5x_3$, s.t. $x_1+x_2+x_3 \leq 7$, $x_1+2x_2+2x_3 \leq 13$, $3x_1-x_2+x_3 \leq 5$, and $x_1, x_2, x_3 \geq 0$.
10. A diet conscious house wife wishes to ensure certain minimum intake of vitamins A, B and C for the family. The minimum daily (quantity) needs of the vitamines A, B and C for the family are respectively 30, 20 and 16 units. For the supply of these minimum vitamin requirements, the house-wife relies on two fresh foods. The first one privides 7, 5, 2 units of the three vitamins per gram respectively, and the second one provides 2,4,8 units of the same three vitamines per gram of the food stuff respectively. The first food stuff costs Rs. 3 per gram and the second Rs. 2 per gram. The problem is how many grams of each food stuff should the house wife buy every day to should the house wife buy every day to keep her food bill as low as possible ?
    (i) Formulate the underlying L.P. problem (ii) Write the 'Dual' problem (iii) Solve the 'Dual' problem by using simplex method. (iv) Solve the primal problem graphically. (v) Interpret the dual problem and its solution.
    [**Ans.** (i) Min. $z=3x_1+2x_2$, s.t. $7x_1+2x_2 \geq 30$, $5x_1+4x_2 \geq 20$, $2x_1+8x_2 \geq 16$, and $x_1, x_2, x_3 \geq 0$.
    (ii) Max. $z_w=30w_1+20w_2+16w_3$, s.t. $7w_1+5w_2+2w_3 \leq 3$, $2w_1+4w_2+8w_3 \leq 2$, and $w_1, w_2, w_3 \geq 0$.
    (iii) $w_1=5/13$, $w_2=0$, $w_3=2/13$, $z_w=14$. (iv) $x_1=4$, $x_2=1$, $z=14$.

7.8. MORE WORKED EXAMPLES

Example 15. *Apply simplex method to solve the following:*

$$\text{Max. } z_x=30x_1+23x_2+29x_3\text{, subject to}$$
$$6x_1+5x_2+3x_3 \leq 26,\quad 4x_1+2x_2+6x_3 \leq 7,\ \text{and all } x_j \geq 0.$$

Also read the solution to the dual of the above problem from the final table.

Solution. Introducing slack variables s_1 and s_2, the given problem becomes:

$$\text{Max. } z_x = 30x_1 + 23x_2 + 29x_3 + 0s_1 + 0s_2 \text{, subject to}$$

$$6x_1 + 5x_2 + 3x_3 + s_1 \qquad = 26$$

$$4x_1 + 2x_2 + 5x_3 \qquad + s_2 = 7,$$

and $\quad x_1, x_2, x_3, s_1, s_2 \geq 0$.

We now give below the successive tables for the solution of this problem by simplex method. The students are advised to verify themselves.

Simplex Table

		$c_j \rightarrow$	30	23	29	0	0	
Basic Var.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{S_1}$	$\mathbf{S_2}$	**Min. Ratio ($\mathbf{X_B/X_1}$)**
s_1	0	26	6	5	3	1	0	26/6
s_2	0	7	$\boxed{4}$	2	5	0	1	7/4 ←
	$z_x = \mathbf{C_B X_B} = 0$		− 30 ↑	$\boxed{-23}$	− 29	0	0 ↓	← Δ_j
s_1	0	31/2	0	2	− 9/2	1	− 3/2	31/4
x_1	30	7/4	1	1/2	5/4	0	1/4	7/2 ←
	$z_x = \mathbf{C_B X_B} = 105/2$		0 ↓	− 8 ↑	17/2	0	15/2	← Δ_j
s_1	0	17/2	− 4	0	− 19/2	1	− 5/2	
x_2	23	7/2	2	1	5/2	0	1/2	
	$z_x = \mathbf{C_B X_B} = 161/2$		16	0	57/2	0	23/2	← Δ_j

Thus, the optimal solution from the final table is given by

$$x_1 = 0,\ x_2 = 7/2,\ x_3 = 0\ ;\ z_x{}^* = 161/2.$$

To read the solution to the dual from the final table :

According to the rules as given in **Section 5·7·2**, the optimal solution to the dual of above problem will be

$$w_1 = \Delta_4 = 0,\ w_2 = \Delta_5 = 23/2,\ z_x{}^* = z_w{}^* = 161/2.$$

In this way, we can find the solution to the dual without actually solving it.

Example 16. *Use duality to solve the problem : Min.* $z_x = x_1 - x_2$, *subject to*

$$2x_1 + x_2 \geq 2,\ -x_1 - x_2 \geq 1, \quad \textit{and} \quad x_1, x_2 \geq 0.$$

Solution. Obviously, the dual of given problem will be of the form :

$$\text{max. } z_w = 2w_1 + w_2 \text{, subject to } 2w_1 - w_2 \leq 1,\ w_1 - w_2 \leq -1 \text{ and } w_1, w_2 \geq 0.$$

We shall have lesser computational efforts in solving the dual rather than the original one. So first, we shall express the dual in standard simplex form :

$$\text{Max. } z_w = 2w_1 + w_2 + 0s_1 + 0s_2 \text{, subject to}$$

$$2w_1 - w_2 \quad + s_2 \qquad = 1$$

$$-w_1 + w_2 - s_1 \qquad + a_1 = 1.$$

We now apply two-phase method to solve it by simplex method.

Phase 1. ***To remove artificial variable*** $\mathbf{a_1}$, *i.e., making* a_1 *non-basic*

$\mathbf{W_B}$	$\mathbf{W_1}$	$\mathbf{W_2}$	$\mathbf{S_1}$	$\mathbf{S_2}$	$\mathbf{A_1}$
1	2	− 1	0	1	0
1	− 1	1	− 1	0	1
		↑			↓
2	1	0	− 1	1	1
1	− 1	1	− 1	0	1
					↓ **Delete**

Phase II. ***To find optimal solution.***

We shall obtain the following successive tables during the simplex procedure.

Basic Var.	C_B	$c_j \rightarrow$ W_B	2 W_1	1 W_2	0 S_1	0 S_2	Min. Ratio (W_B/W_1)
s_2	0	2	$\boxed{1}$	0	-1	1	$(2/1) \leftarrow$
w_2	1	1	-1	1	-1	0	×
	$z_w = C_B W_B = 1$		-3 ↑	0	-1	0 ↓	$\leftarrow \Delta_j$
w_1	2	2	1	0	-1	1	
w_2	1	3	0	1	-2	1	
	$z_w{}^* = C_B W_B = 4$		0	0	-4	3	$\leftarrow \Delta_j$

$$\Delta_4 = C_B S_2 - c_4 = (2\ ,\ 1)\ (1\ ,\ 1) - 0 = 3,\ \Delta_3 = C_B S_1 - c_3 = (2\ ,\ 1)\ (-1\ ,\ -2) - 0 = -4\ .$$

Since Δ_3 is –ve and all elements of vector S_1 (not in the basis) are negative, this indicates that the problem under solution has unbounded solution. Consequently, by *duality theorem*, the original primal problem will have no feasible solution.

Example 17. *Use duality to solve* : *min.*$z_x = 3x_1 + x_2$, *subject to* $x_1 + x_2 \geq 1$, $2x_1 + 3x_2 \geq 2$, *and* $x_1\ , x_2 > 0$.

Solution. The dual of given problem will be of the form :

$$\text{Max.}z_w = 1w_1 + 2w_2\ ,\ \text{subject to}\ w_1 + 2w_2 \leq 3,\ w_1 + 3w_2 \leq 1,\ \text{and}\ w_1\ ,\ w_2 \geq 0.$$

This problem can be easily solved by simplex method. Introducing the slack variables s_1 and s_2 and proceeding in the usual simplex routine, we get the following successive tables.

Simplex Table

Basic Variables	C_B	$c_j \rightarrow$ W_B	1 W_1	2 W_2	0 S_1	0 S_2	Min. Ratio (W_B/W_k)
s_1	0	3	1	2	1	0	3/2
s_2	0	1	1	$\boxed{3}$	0	1	$1/3 \leftarrow$
	$z_w = C_B W_B = 0$		-1	-2 ↑	0	0 ↓	$\leftarrow \Delta_j$
s_1	0	7/3	1/3	0	1	$-2/3$	7/1
w_2	2	1/3	$\boxed{1/3}$	1	0	1/3	$1 \leftarrow$
	$z_w = C_B W_B = 2/3$		$-1/3$ ↑	0 ↓	0	2/3	$\leftarrow \Delta_j$
s_1	0	2	0	-1	1	-1	
w_1	1	1	1	3	0	1	
	$z_w{}^* = C_B W_B = 1$		0	1	0	1	$\leftarrow \Delta_j$

Hence the optimal solution of the dual comes out to be $w_1 = 1\ \ w_2 = 0$, with $z_w^* = 1$.

To read the solution of original primal from the final table : $x_1 = \Delta_3 = 0$, $x_2 = \Delta_4 = 1$, $z_w^* = z_x^* = 1$.

EXAMINATION PROBLEMS

Use duality in obtaining an optimal solution, if any, to each of the following linear programming problems :

1. Max. $z = 8x_1 + 6x_2$, subject to the constraints : $x_1 - x_2 \leq 3/5$, $x_1 - x_2 \geq 2$, and $x_1, x_2 \geq 0$.
   [**Ans.** Dual problem does not possess a feasible solution]
2. Max. $z = 2x_1 + x_2$, subject to the constraints : $x_1 + x_2 \geq 2$, $x_1 + 3x_2 \leq 3$, and $x_1, x_2 \geq 0$.
   [**Ans.** $x_1 = 4$, $x_2 = 2$; max $z = 10$]

3. Max. $z = x_1 + 6x_2$, subject to the constraints : $x_1 + x_2 \geq 2$, $x_1 + 3x_2 \leq 3$; $x_1, x_2 \geq 0$.
[**Ans.** $x_1 = 3/2$, $x_2 = 1/2$; max. $z = 9/2$]
4. Max. $z = 30x_1 + 24x_2$, subject to the constraints : $x_1 + 2x_2 \geq 80$, $-2x_1 + x_2 \leq 10$, $x_1 - x_2 \geq 30$, $x_1, x_2 \geq 0$
[**Ans.** Unbounded solution]
5. Max. $z = 2x_1 + x_2$, subject to the constraints : $x_1 + 2x_2 \leq 10$, $x_1 + x_2 \leq 6$, $x_1 - x_2 \leq 2$, $x_1 - 2x_2 \leq 1$, $x_1 \geq 0$, $x_2 \geq 0$.
[**Ans.** $x_2 = 4$, $x_2 = 2$, max. $z = 10$] **[Tamil. (ERON) 97; Kerala B.Sc. 90]**
6. Min. $z = -2x_1 + 3x_2 + 4x_3$, subject to the constraints : $-2x_1 + x_2 \geq 3$, $-x_1 + 3x_2 + x_3 \geq -1$, $x_1, x_2, x_3 \geq 0$
[**Ans.** $x_1 = 0$, $x_2 = 3$, $x_3 = 0$, min. $z = 9$]
7. Min. $z = 4x_1 + 3x_2 + 3x_3$, subject to the constraints : $x_1 + 2x_2 \geq 2$, $3x_1 + x_2 + x_3 \geq 4$, $4x_3 \geq 1$, $x_1 + x_3 \geq 1$, $x_1, x_2, x_3 \geq 0$.
8. Write down the primal problem to the following dual problem :
Max. $W_1 + W_2 + W_3$, subject to $2W_1 + W_2 + 2W_3 \leq 2$, $4W_1 + 2W_2 + W_3 \leq 2$, $W_1, W_2, W_3 \geq 0$.
[VTU 2002]

7.9. SHADOW PRICES IN LINEAR PROGRAMMING

Let us consider the primal problem : Maximize $z = \mathbf{CX}$, subject to the constraints $\mathbf{AX} = \mathbf{b}$; $\mathbf{X} \geq \mathbf{0}$, where $\mathbf{X}$ and $\mathbf{C} \in R^n$, $\mathbf{b} \in R^m$ and $\mathbf{A}$ is an $m \times n$ real matrix. Also, let $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$ be an optimum basic feasible solution to this primal problem, where $\mathbf{B}$ is the optimal basis matrix. If $\mathbf{C_B}$ denotes the cost vector associated with basic variables, then the optimum value of the objective function will be $z^* = \mathbf{C_B X_B} = \mathbf{C_B}(\mathbf{B}^{-1}\mathbf{b})$.

Now we may define the *shadow* (or *implicit* or *marginal*) price p_i of the ith resource b_i (right hand side value) to be the achievable rate of increase in resource i, as follows :

$$\frac{dz^*}{db_i} = c_{Bi}\,\mathbf{B}^{-1} \equiv \mathbf{X}_i^*$$

where $\mathbf{X}^* = \mathbf{C_B}\,\mathbf{B}^{-1} = \sum_{i=1}^{m} c_{Bi}\,\hat{\beta}_{ik}$, $\hat{\beta}_{ik}$ being the kth column vector of $\mathbf{B}^{-1}$.

Thus, $\mathbf{X}_i^*$ is the rate of change of the optimal objective function value with respect to b_i and is called the *optimal dual variable* or *simplex multipliers*.

For several practical purposes, the calculation of shadow prices is more important than the solution of the problem, because it permits the user to ensure whether certain potential changes in the model requirements might actually increase the objective function.

For example, if one of the resources of a company represent the current production capacity of a particular plant and the shadow price for this resource is greater than the actual unit cost increasing the capacity of the plant, then the company could increase its profit by doing so.

Example 18. *A firm makes two products A and B. Each product requires production* on each of the two machines :

Machine	Product		Available (in hours)
	A	B	
M_1	6	4	60
M_2	1	2	22

Total time available is 60 hours and 22 hours on machines M_1 and M_2 *respectively. Product A and B contribute Rs. 3 and Rs. 4 per unit respectively.*

Determine the optimum product mix. Write the dual of this problem and give its economic interpretation.

Solution. ***Formulation of Primal and Dual Problems :***

Primal Problem : Maximize $z_x = 3x_1 + 4x_2$, subject to the constraints :

$$6x_1 + 4x_2 \leq 60, \quad x_1 + 2x_2 \leq 22, \quad \text{and} \quad x_1 \geq 0, x_2 \geq 0,$$

where x_1 is the number of units of product A and x_2 is the number of units of product B.

Dual Problem : Minimize $z_w = 60\,w_1 + 22w_2$, subject to the constraints :

$$6w_1 + w_2 \geq 3, \quad 4w_1 + 2w_2 \geq 4, \quad \text{and} \quad w_1 \geq 0, \; w_2 \geq 0,$$

where w_1 is the cost per hour on machine M_1, and w_2 is the cost per hour on machine M_2.

Optimum Solution to the Primal :

By usual simplex mathod, the optimum simplex table is given ahead :

		$c_j \rightarrow$	3	4	0	0
Basic Var.	C_B	X_B	X_1	X_2	X_3	X_4
x_1	3	4	1	0	1/4	– 1/2
x_2	4	9	0	1	– 1/8	– 3/4
	$z = C_B X_B = 48$		0	0	1/4	3/2

Thus the optimum solution is : $x_1 = 4$, $x_2 = 9$ and max. profit = Rs. 48.

Using duality, optimum solution to the dual problem can be read directly from the optimum simplex table as given by : w_1 = Rs. 0.25 per hour, w_2 = Rs. 1·50 per hour, and min. cost = Rs. 48.

[IAS (Maths) 99]

Economic Interpretation of Dual :

As discussed above, shadow prices are the opportunity costs that indicate the potential profit that is lost by not having an additional unit of the respective right hand side (resource) assuming that all right hand side values are used optimally. Thus $w_1 = 0{\cdot}25$ and $w_2 = 1{\cdot}50$ means that additional processing hours on machine M_1 and M_2 will increase the profit by Rs. 0·25 and Rs. 1·50, respectively.

Similarly in the primal problem, if we increase the total available hours on machine M_1 from 60 hours to 61 hours, the new set of constraints will be : $6x_1 + 4x_2 \leq 61$ and $x_1 + 2x_2 \leq 22$.

Solving the primal problem with new set of constraints, the optimum solution becomes :

$$x_1 = 4{\cdot}25 \text{ and } x_2 = 8{\cdot}875 \text{ with max } z = 48{\cdot}25$$

This is exactly Rs. (0·25)/1 = Re 0·25 more than the earlier value of z when only 60 hours of machine M_1 were available.

Q. 1. Write a note on dual prices and marginal valuation.
2. Explain briefly the economic significance of duality.
3. Give the economic interpretation of the dual problem assuming the primal programme to be a standard production problem. **[IAS (Math) 99]**
4. Discuss the dual of a diet problem and give its economic interpretation.

7·10. ADVANTAGES OF DUALITY

The knowledge of the dual is important for the following main reasons :

(i) As seen from above worked examples, the solution may be easier to obtain through its dual than through the original (primal) problem. This is true of cases in which the number of original variables in the primal problem is considerably less than the number of slack or surplus variables.

(ii) Duality is not only restricted to linear programming problems but frequently occurs in economics, physics, engineering, mathematics and other fields also.

(a) In *economics,* it is used in the formulation of the input and output systems. The economic interpretation of the dual is found useful in making future decisions in the activities being programmed.

(b) In *physics,* it is used in the parallel circuit and series circuit theory.

(iii) In game theory, it is used to find the optimal strategies of the other player *B* when he minimizes his losses. Then, by duality, we can change the player *A*'s problem into player *B*'s problem and *vice-versa.* If we solve the problem for one player, the solution to the other can be easily read-off by using duality properties.

Q. 1. Explain the significance of duality theory in linear programming. For a given LP problem state its dual and show that the dual of the dual is the primal.
2. What are the useful aspects of duality in L.P.P.

SELF-EXAMINATION QUESTIONS

1. Define the dual of L.P.P. State the fundamental properties of duality and prove any one of them.
2. Let $\mathbf{x_0}$ be any solution to the L.P.P. of maximizing $z = \mathbf{cx}$ subject to the constraints $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq 0$. If $\mathbf{w_0}$ is any solution of its dual, then show that $\mathbf{w_0 x_s^0} = 0$, where $\mathbf{x_s^0}$ is the slack vector corresponding to $\mathbf{x_0}$.

3. Write a short note on duality theory.
4. Let $\tilde{x}$ be a feasible solution to the primal problem : Max. $\mathbf{c}'\mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b}$, $\mathbf{x} \ge \mathbf{0}$; and $\tilde{\mathbf{y}}$ feasible for its dual. Prove that Max. $\mathbf{c}'\tilde{\mathbf{x}} \le$ Min. $\mathbf{b}'\mathbf{y}$.
If $\mathbf{x}*$ and $\mathbf{y}*$ be feasible solutions and $\mathbf{c}'\mathbf{x}* = \mathbf{b}'\mathbf{y}*$, what is your conclusion ?
5. Prove that if either the primal or the dual problem has a finite optimum solution, then the other problem has a finite optimum solution and the extreme of the linear functions are equal, i.e. min. f = max. g. And hence show that if either problem has an unbounded optimum solution, then the other problem has no feasible solution.
6. State and prove (i) weak duality theorem (ii) basic duality theorem and (iii) the fundamental duality theorem.
7. Consider the following problems : P_1 : Max $\mathbf{CX}$ such that $\mathbf{AX} \le \mathbf{b}$, $\mathbf{X} \ge 0$; and P_2 : Min $\mathbf{Wb}$ such that $\mathbf{WA} \ge \mathbf{C}$, $\mathbf{W} \ge 0$.
   (a) If $\mathbf{X}^0$ and $\mathbf{W}^0$ are feasible for P_1 and P_2 respectively, show that
      (i) $\mathbf{CX}^0 \le \mathbf{W}^0\mathbf{b}$
      (ii) If $\mathbf{CX}^0 = \mathbf{W}^0\mathbf{b}$, then $\mathbf{X}^0$ and $\mathbf{W}^0$ are optimal for P_1 and P_2 respectively.
      (iii) $W^0(\mathbf{AX}^0 - \mathbf{b}) = 0$ (iv) $(W^0A - C)X^0 = 0$.
   (b) Show that $\mathbf{X}^0$ and $\mathbf{Y}^0$ are feasible for P_1 and P_2 respectively such that (iii) and (iv) in (a) are satisfied, then $\mathbf{X}^0$ and $\mathbf{W}^0$ are optimal for the respective problems.
8. If $\mathbf{X}$ is any feasible solution of the linear programming problem : Maximize $f(x) = \mathbf{CX}$ subject to $\mathbf{AX} \le \mathbf{b}$, $\mathbf{X} \ge 0$ fielding the value $f(\mathbf{X})$ of the objective function and $\mathbf{Y}$ be any feasible solution of its dual problem yielding the value $g(\mathbf{y})$ of the dual objective function, then show that $f(\mathbf{X}) \le g(\mathbf{Y})$. Further show that if the primal and dual problems have feasible solutions $\mathbf{X}$ and $\mathbf{Y}$, respectively, such that $f(\mathbf{X}) = g(\mathbf{Y})$, then $\mathbf{X}$ is optimal for the primal and $\mathbf{Y}$ is optimal for the dual.

[Delhi B.Sc. (Maths.) 93]

EXAMINATION PROBLEMS

1. Prove that the dual of the dual is the primal. Write down the dual of the following problems, and solve them.

(i) Max. $z + 3x_1 + x_2 + x_3 - x_4$
s.t. $x_1 + 5x_2 + 3x_3 + 4x_4 \le 5$
$x_1 + x_2 = -1$
$x_3 - x_4 \le -5$
$x_j \ge 0$, $j = 1, 2, 3, 4$.

(ii) Max. $z = 4x_1 + 2x_2$,
s.t. $x_1 + x_2 \ge 3$
$x_1 - x_2 \ge 2$
$x_1 , x_2 \ge 0$.

Hence or otherwise write down the solution of the above primal problems.

7. Use duality to obtain an optimum solution, if any, to the following linear programming problems.

(i) Max. $z + 2x_1 + 3x_2$,
subject to
$-x_1 + 2x_2 \le 4$
$x_1 + x_2 \le 6$
$x_1 + 3x_2 \le 9$
$x_1 , x_2 > 0$.
[Ans. $x_1 = 9/2$, $x_2 = 3/2$, max. $z = 27/2$].

(ii) Min. $z = 15x_1 + 10x_2$
subject to
$3x_1 + 5x_2 \ge 5$
$5x_1 + 2x_2 \ge 3$
$x_1 , x_2 \ge 0$.
[Ans. $x_1 = 5/19$, $x_2 = 16/19$, min. $z = 235/19$].

(iii) By means of duality theory, solve and illustrate geometrically the following L.P.P. :
Max. $z = 3x_1 + 2x_2$
subject to
$x_1 + x_2 \ge 1$
$x_1 + x_2 \le 7$
$x_1 + 2x_2 \le 10$
$x_2 \le 3$
$x_1 , x_2 \ge 0$.
[**Ans.** $x_1 = 4$, $x_2 = 3$, max. $z = 18$].

(iv) Max. $z = 6x_1 + 4x_2 + 6x_3 + x_4$
subject to
$4x_1 + 5x_2 + 4x_3 + 8x_4 = 21$
$3x_1 + 7x_2 + 8x_3 + 2x_4 \le 48$
$x_1 , x_2 , x_3 , x_4 \ge 0$.
[**Ans.** $x_1 = 21/4$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, max. $z = 63/2$]

(v) Max. $\mathbf{R} = 6x + 5y - 3z - 4w$
subject to
$2x + 3y + 2z - 4w = 24$
$x + 2y \le 10$
$x + y + 2z + 3w \le 15$
$y + z + w \le 8$.
[**Ans.** Feasible solution does not exist].

(vi) Max. $z = 4x_1 + 3x_2$
subject to
$x_1 \le 6$
$x_2 \le 8$
$x_1 + x_2 \le 7$
$3x_1 + x_2 \le 15$
$-x_2 \le 1$
$x_1 , x_2 \ge 0$.
[**Ans.** $x_1 = 4$, $x_2 = 3$, max $\mathbf{z} = 25$]

(vii) Max. $z = x_1 + 5x_2$
subject to
$3x_1 + 4x_2 \le 6$
$x_1 + 3x_2 \le 2$
$x_1, x_2 \ge 0.$
[**Ans.** Problem has no solution].

(viii) Min. $z = x_1 + x_2$,
subject to
$2x_1 + x_2 \ge 4$
$x_1 + 7x_2 \ge 7$
$x_1, x_2 \ge 0.$

(ix) Solve the dual of the following problem graphically.
Min. $z = 10y_1 + 6y_2 + 2y_3$
s.t. $-y_1 + y_2 + y_3 \ge 1$
$3y_1 + y_2 - y_3 \ge 2$
$y_1, y_2, y_3 \ge 0$.
[**Ans.** $x_1 = 1/4$, $x_2 = 5/4$, $x_3 = 0$, Min $z = 10$.]

(x) $3x_1 + 5x_2 + 4x_3 \ge 7$
$6x_1 + x_2 + 3x_3 \ge 4$
$7x_1 - 2x_2 - x_3 \le 10$
$4x_1 + 7x_2 - 2x_3 \ge 2$
$x_1 - 2x_2 + 5x_3 \ge 3$
Min. $z = 3x_1 - 2x_2 + 4x_3$,
$x_1, x_2, x_3, x_4 \ge 0$.

3. Solve the following problem by simplex method.
$3x_1 - x_2 - x_3 \ge 3$, $x_1 - x_2 + x_3 \ge 2$ and $x_1, x_2, x_3 \ge 0$, $z = (15/2)\,x_1 - 3x_2$ to be minimum.
Write the dual of the above problem. What should be the maximum value of the objective function of the dual ?

4. Consider the problem : Max. $z = 2x_1 + 3x_2$, subject to
$2x_1 + 2x_2 \le 10$, $2x_1 + x_2 \le 6$, $x_1 + 2x_2 \le 6$, and $x_1, x_2 \ge 0$.
(a) Write the complete dual problem from the canonical form of the above primal.
(b) Solve the primal problem and then find the solution to the dual.

5. Find the dual of the problem :
Max. $f(\mathbf{x}) = 2x_1 - x_2$, subject to the constraints : $x_1 + x_2 \le 10$, $-2x_1 + x_2 = 2$, $4x_1 + 3x_2 \ge 12$; $x_1, x_2 \ge 0$.
Solve the primal problem by simplex method and deduce from it the solution to the dual problem.

6. Use duality theory to solve the following linear programming problem :
Min. $z = 4x_1 + 3x_2 + 6x_3$, s.t. $x_1 + x_3 \ge 2$, $x_2 + x_3 \ge 5$; $x_1, x_2, x_3, \ge 0$.

7. Find the dual of the following problem and hence or otherwise solve it.
Min. $f(\mathbf{x}) = 6x + 5y - 2z$, subject to $x + 3y + 2z \ge 5$, $2x + 2y + z \ge 2$, $4x - 2y + 3z \ge -1$, and $x_1, x_2, < x_3 \ge 0$

8. A company makes three products X, Y, Z out of three materials P_1, P_2 and P_3. The three products use units of the three materials according to the following table :

	P_1	P_2	P_3
X	1	2	3
Y	2	1	1
Z	3	2	1

The unit profit contribution of the three products are :

Product	:	X	Y	Z
Profit contribution (Rs.)	:	3	4	5

and availabilities of the three materials are :

Material	:	P_1	P_2	P_3
Amount Available (units)	:	10	12	15

The problem is to determine the product mix,. which will maximize the total profit.
Solve the primal problem and wirte the dual and give geometrical interpretation.

9. A firm makes three products A, B and C. Each product requires production time in each of three departments as shown below :

Products	*Time taken (in hours per unit)*		
	Deptt. I	*Deptt. II*	*Deptt. III*
A	3	2	1
B	4	1	3
C	2	2	3

Total time available is 60 hours, 40 hours, and 30 hours in departments I, II and III respectively. If product A contriutes Rs. 2 per unit and product B and C Rs. 4 and Rs. 2.50 respectively, determine the optimum product mix.
Write the dual of this problem and give its economic interpretation.

10. Consider the problem : Max. $z = 8x_1 + 6x_2$, s.t. $x_1 - x_2 \le 3/5$, $x_1 - x_2 \ge 2$, and $x_1, x_2 \ge 0$
Show that both the primal and the dual problem have no feasible solution

11. Consider problem A : Min. $z = x_1 - 10x_2$, s.t. $x_1 - \cdot 5x_2 \ge 0$, $x_1 - 5x_2 \ge -5$, and $x_1, x_2 \ge 0$
and problem B : Max $z = -5x_2$, s.t. $x_1 + x_2 \le 1$, $-\cdot 5x_1 - 5x_2 \le -10$, and $x_1, x_2 \ge 0$.
Explain, how the solutions of A and B are related ?

12. Find the dual of the problem : Max. $z = -x_1 + 2x_2 - x_3$
s.t. $3x_1 + x_2 - x_3 \le 10$, $-x_1 + 4x_2 + x_3 \ge 6$, $x_2 + x_3 \le 4$, $x_1, x_2, x_3 \ge 0$.
Solve the primal by simplex method and deduce the solution of the dual problem from the optimal table of the primal.

13. A dairy has two bottling plants one located at A, and other at B. Each plant bottles up three different kinds of milk, *i.e.* Cow's Toned and Double Toned. The capacities of the two plants in number of bottles per shift in a day are as follows :

Milk	Plant	
	A	B
Cow's	2,000	1,000
Toned	2,000	3,000
Double Toned	1,000	1,000

Market survey shows that the demand of Cow's, Toned, and Double Toned milk are at least 14,000, 22,000 and 1,000 bottles per day. The operating costs per shifts of running plants A and B are respectively Rs. 900 and Rs. 600 only. How many shifts should the firm run each plant per day so that the production cost is minimum while still meeting the markets demand.

Write the dual of this and give an economic interpretation of the dual variables.

14. The *XYZ* company has the option of producing two products during the period of slack activity. For the next period, production has been scheduled so that the milling machine is free for 10 hours and skilled labour will have 8 hours of time available.

Product	Machine time per unit	Skilled labour time per unit	Profit contribution per unit (Rs.)
A	4	2	5
B	2	2	3

Solve the primal and dual problem by the simplex method and bring out the fact that the optimum solution of one can be obtained from the other. Also explain the context of the example what you understand by shadow prices (or dual prices or marginal value) of resource.

(Jammu (M.B.A.) Nov. 96)

[**Hint :** The primal problem is : Max. $z = 5x_1 + 3x_2$, *s.t.* $4x_1 + 2x_2 \le 10$, $2x_1 + 2x_2 \le 8$; $x_1, x_2 \ge 0$

Ans. $x_1 = 1$, $x_2 = 3$ and Max. $z = 14$.

The dual problem is : Min. $z = 10y_1 + 8y_2$, *s.t.* $4y_1 + 2y_2 \ge 5$, $2y_1 + 2y_2 \ge 3$; $y_1, y_2 \ge 0$

Ans. $y_1 = 1$, $y_2 = 1/2$ and Min. $z = 14$.]

15. A cabinet manufacturer produces wood cabinets for TV sets, Stereosystems and radios, each of which must be assembled, decorated and crated. Each TV cabinet requires 3 hours to assemble, 5 hours to decorate and 1/10 hour to crate and returns a profit of Rs. 10. Each stereo cabinet requires 10 hours to assemble, 8 hours to decorate and 3/5 hour to crate and return a profit of Rs. 25. Each radio cabinet requires 1 hour to assemble, 1 hour to decorate and 1/10 hour to crate and returns a profit of Rs. 3. The manufacturer has the maximum of 30,000, 40,000 and 120 hours available for assembling, decorating and crating respectively. You are required to :
(i) Formulate the above problem as linear programming problem.
(ii) Use simplex method to find how many units of each product should be manufactured to maximize profit.
(iii) Write the dual of the problem and find its solution from the primal.
(iv) Has the problem multiple solutions ?

(Delhi (M.B.A.,) Nov. 97)

16. A person consumes two types of food *A* and *B* everyday to obtain 8 units of proteins, 12 units of carbohydrates and 9 units of fats which is his daily minimum requirements. 1 kilo of food *A* contains 2, 6 and 1 units of protein, carbohydrates and fats respectively. 1 kilo of food *B* contains 1, 1 and 3 units of proteins, carbohydrates and fats respectively. Food *A* costs Rs. 8.50 per kilo, while *B* costs Rs. 4 per kilo. Determine how many kilos of each food should he buy daily to minimize his cost of food and still meet the minimum requirements.

Formulate the problem mathematically. Write its dual and solve the dual by the simplex method.

(Gujarat (M.B.A.) Feb. 96)

[**Hint :** Min $z = 8{\cdot}50x_1 + 4x_2$, *s.t.* $2x_1 + x_2 \ge 8$, $6x_1 + x_2 \ge 12$, $x_1 + 3x_2 \ge 9$; $x_1, x_2 \ge 0$.

Ans. $x_1 = 1$, $x_2 = 6$, min $z = 65/2$.

The Dual problem is :

Max. $z' = 8y_1 + 12y_2 + 9y_3$, *s.t.* $2y_1 + 6y_2 + y_3 \le 8{\cdot}50$, $y_1 + y_2 + 3y_3 \le 4$, $y_1, y_2, y_3 \ge 0$.

Ans. $y_1 = 31/8$, $y_2 = 1/8$, $y_3 = 0$, max. $z' = 65/2$.]

17. Three food products are available at the cost of Rs. 10, Rs. 36 and Rs. 24 per unit, respectively. They contain 1,000, 4,000 and 2,000 calories per unit, respectively, and 200, 900 and 500 protein units per unit, respectively. It is required to find the minimum cost diet containing at least 20,000 calories and 3,000 units of protein. Formulate and solve the given problem. Write the dual and use it to check the optimum solution of the given problem. **(Shivaji (M.B.A.) 95)**

18. A company has Rs. 1,80,000 available for purchase of machines to produce a new product. The manufacturer of the machines offers models : the regular model, costing Rs. 3,000; the deluxe model, costing Rs. 5,000; and the super model, costing Rs. 10,000. The regular model occupies 15 square feet of floor space, requires one operator to run, and can produce 4 units of product per hour. The deluxe model occupies 20 square feet of floor space, requires two operators to run, and can produce 5 units of product per hour. The super model occupies 30 square feet of floor space, requires five operators to run, and can produce 20 units of product per hour. The company insists that at least half of the machines purchased be the deluxe model. A unit of product is expected to have gross profit of Rs. 10, excluding the labour cost of

the operators. It is felt that all products can be sold at the Rs. 10 per unit profit figure. The company has available 500 square feet of floor space for the machines and 100 operators to run the machines. Operators are paid Rs. 4 per hour.

(i) How many machines of each type should be purchased in order to maximize profit ?

(ii) Write the dual of the given problem and use it for checking the optimal solution. **(Punjab (M.B.A.) 96)**

19. Respond *True* or *False* to the following, justify your answer in case of False.

(i) If the number of primal variables is much smaller than the number of constraints, it is more difficult to obtain the solution of the primal by solving its dual.

(ii) When the primal problem is non-optimal, the dual problem is automatically infeasiable.

(iii) An unrestricted primal variable will have the effect of yeilding an equality dual constraint.

(iv) If the solution space is unbounded, the objective value always will be unbounded.

(v) The selection of the entering variable from among the current non-basic variables as the one with the most negative objective coefficient guarantees tha most increase in the objective value in the next iteration.

(vi) In the simplex method, the feasibility condition for the maximization and minimization problems are different.

(vii) A simplex iteration (basic solution) may not necessarily coincide with a feasiable extreme point of the solution space.

(viii)If the leaving variable does not correspond to the minimum ratio, at least one basic variable will definitely become negative in the next iteration. **(IAS (Maths.) 99)**

20. Using duality or otherwise solve the linear programming problem :

Minimize $18x_1 + 12x_2$

Subject to $2x_1 - 2x_2 \geq -3$

$3x_1 + 2x_2 \geq 3$

$x_1 , x_2 \geq 0$. **[IAS (Main) 2001]**

21. Explain the duality principle in linear programming. Construct the dual of the problem :

$$\begin{array}{ll} \text{Maximize} & z = 3x_1 + 17x_2 + 9x_3 \\ \text{subject to} & x_1 - x_2 + x_3 \geq 3 \\ & -3x_1 + 2x_3 \leq 1 \\ & 2x_1 + x_2 - x_3 = 4 \\ \text{with} & x_1, x_2, x_3 \geq 0. \end{array}$$

[AIMS (Bang.) MBA 2002]

22. Solve the following problem by dual method : Maximize $z = 30x_1 + 20\,x_2$, subject to the constraints $-x_1 - x_2 \geq -8$, $-6x_1 - 4x_2 \leq -12$, $5x_1 + 8x_2 = 20$, $x_1, x_2 \geq 0$, **[JNTU (Mech. & Prod.) 2004]**

OBJECTIVE QUESTIONS

1. The dual of the primal maximization LP problem having *m* constraints and *n* non-negative variables should
(a) have *n* constraints and *m* non-negative variables. (b) be a minimization LP problem.
(c) both (a) and (b). (d) none of the above.

2. For any primal problem and its dual:
(a) optimal value of objective function is same.
(b) primal will have an optimal solution if and only if dual does too.
(c) both primal and dual cannot be infeasible.
(d) all of the above.

3. The right hand side constant of a constraint in a primal problem appears in the corresponding dual as
(a) a coefficient in the objective function. (b) a right hand side constant of a constraint.
(c) an input-output coefficient. (d) none of the above.

4. Dual LP problem approach attempts to optimize resource allocation by ensuring that
(a) marginal opportunity cost of a resource equals its marginal return.
(b) marginal opportunity cost of a resource is less than its marginal return.
(c) both (a) and (b).
(d) none of the above.

5. Shadow price indicates how much one unit change in the resource value will change the
(a) optimality range of an objective function (b) optimal value of the objective function
(c) value of the basic variable in the optimal solution. (d) none of the above

6. Principle of complementary slackness states that
(a) primal slack × dual main = 0. (b) primal main × dual surplus = 0.
(c) both (a) (b). (d) none of the above.

7. If dual has an unbounded solution, primal has
(a) no feasible solution. (b) unbounded solution. (c) feasible solution. (d) none of the above.

8. If at the optimality a primal constraint has positive value of slack variable, then
(a) dual variable corresponding to that constraint has zero value.
(b) corresponding resource is not completely used up.

(c) corresponding resource have zero opportunity cost.
(d) both (b) and (c) but not (a).

9. The shadow price is
(a) the price that is paid for purchase of resources.
(b) the saving by eliminating one of the excess quantities of resource.
(c) the increase in the objective function value by providing one additional unit of resource.
(d) none of the above.

10. The value of dual variable
(a) represents marginal profit of each additional unit of resource
(b) can be obtained by examining the z_j row fo primal optimal simplex table
(c) can be obtained by examining $z_j - c_j$ row of primal optimal simplex table.
(d) all of the above.

Answers

1. (c) **2.** (b) **3.** (a) **4.** (a) **5.** (b) **6.** (c) **7.** (a) **8.** (d) **9.** (c) **10.** (a).

***

8

THE DUAL SIMPLEX METHOD

8.1. INTRODUCTION

In simplex method, we have already seen that every basic solution with all $z_j - c_j \geq 0$ will not be feasible (since $z_j - c_j = \mathbf{c_B B^{-1} a_j} - c_j$ is independent of vector **b** for all j), but any basic feasible solution with all $z_j - c_j \geq 0$ will certainly be an optimal solution. Such type of typical problems, for which it is possible to find *infeasible* but better than optimal initial basic solution (with all $z_j - c_j \geq 0$), can be solved more easily by *dual simplex method.* Such a situation is recognized by first expressing the constraints in the form ($\leq$) and the objective function in the maximization form. After adding the slack variables and putting the problem in the tableau form, if any of the right hand side elements are negative and if the optimality condition is satisfied, then the problem can be solved by the dual simplex method. It is important to note that by this arrangement, negative element on the right hand side signifies that the corresponding slack variable is negative. This means that the problem starts with optimal but infeasible basic solution as required by dual simplex procedure. In this method, we shall proceed towards feasibility maintaining optimality and at the iteration where the basic solution becomes feasible, it becomes the optimal basic feasible solution also.

The dual simplex method is very similar to the regular simplex method. In fact, once they are started, the only difference lies in the criterion used for selecting a vector *to enter the basis* and *to leave the basis.* Also, it is to be noted that in the dual simplex method, we first determine the vector to leave the basis and then the vector to enter the basis. This is just reverse of what is done in the simplex method. The dual simplex method yields an optimal solution to the given linear programming problem (that can be handled by this method) in a finite number of steps, provided no basis had to be repeated.

Since the dual simplex method deals with the primal problem as if the simplex method were being applied simultaneously to its dual problem, and the criteria used for inserting and leaving vectors are those for the dual rather than the primal problem ; that is why this method is called the *dual simplex method.*

Before giving the details of dual simplex method, we shall discuss in the following section how an optimal but infeasible solution can be obtained to start with.

8.2. COMPUTATIONAL PROCEDURE OF DUAL SIMPLEX METHOD

8.2-1. To find the initial solution which is infeasible but optimal

As already pointed out, it is difficult to find, in general, a starting basic solution to the primal with all $z_j - c_j \geq 0$ (*i.e.*, optimal). However, it is easy to obtain such an initial basic solution for typical type of problems only (for such problems the dual simplex method is applicable).

First of all, we should remember that we can find the initial basic solution to the primal with all $z_j - c_j \geq 0$ (optimal) only when $c_j \leq 0$ for every j in the maximization objective function. In other words, the coefficients of the variables in the maximization objective function must be non-positive in order to find an initial solution to start with dual simplex method.

To illustrate the procedure of finding an infeasible but optimal solution to start with dual simplex method, let us consider the following simple example :

Example 1. *Find an infeasible but optimal basic solution for the linear programming problem :*
Max. $z = -4x_1 - 6x_2 - 18x_3$, *subject to* $x_1 + 3x_3 \geq 3,\ x_2 + 2x_3 \geq 5$, and $x_1, x_2, x_3 \geq 0$.

Solution. Step 1. First of all, we judge whether it is possible or not to find an infeasible but optimal basic solution.

Since the objective function is that of maximization and all c_j's are negative ($c_1 = -4$, $c_2 = -6$, $c_3 = -18$), it is possible to find such a starting solution.

Now, introducing surplus variables x_4 and x_5 to each constraint, we have

$$\begin{aligned} x_1 \quad + 3x_3 - x_4 \quad &= 3 \\ x_2 + 2x_3 \quad - x_5 &= 5 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 . \end{aligned}$$

In order to get initial basis matrix as an identity matrix we multiply the constraint equations through by -1 and get

$$\begin{aligned} -x_1 \quad - 3x_3 + x_4 \quad &= -3 \\ -x_2 - \quad 2x_3 \quad + x_5 &= -5 . \end{aligned}$$

We now construct the starting simplex *Table* 8·1 in usual form. In this table,

$$\Delta_1 = \mathbf{C_B X_1} - c_1 = 4 , \Delta_2 = \mathbf{C_B X_2} - c_2 = 6 , \Delta_3 = \mathbf{C_B X_3} - c_3 = 18 .$$

Thus, the solution from the *Table* 8·1 is given by $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = -3$, $x_5 = -5$, which is obviously infeasible but optimal (since all Δ_j's being ≥ 0). We shall always start the dual simplex method with such an initial solution.

Table 8·1

		$c_j \rightarrow$	-4	-6	-18	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	
x_4	0	-3	-1	0	-3	1	0	
x_5	0	-5	0	-1	-2	0	1	
	$z = \mathbf{C_B X_B} = 0$		4	6	18	0	0	$\leftarrow \Delta_j$

Like the regular simplex method, the method of solution requires two conditions, the *optimality* and the *feasibility*. The optimality condition ensures that the solution remains optimal all the time while the feasibility condition forces the basic solutions towards the non-negativity of basic variables.

The *criteria for leaving vector* ensures the feasibility condition while the *criteria for entering vector* ensures the optimality condition.

We now proceed to find the *'leaving vector'* and the *'entering vector'* in the following steps.

Step 2. To find the vector (β_r) to leave the basis

We always remove the vector β_r for which r is obtained by

$$x_{Br} = \min_i \, [x_{Bi} , x_{Bi} < 0] . \qquad ...(8\cdot1)$$

In above example, we have $\quad x_{Br} = \min (x_{B1} , x_{B2}) = \min (-3 , -5) = -5 = x_{B2}$

Therefore, $r = 2$. So we must remove the vector β_2, *i.e.* $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Thus, the leaving vector is selected corresponding to the basic variable having the most negative value. If all the basic variables are non-negative, the process ends and the feasible (optimal) solution is reached.

Step 3. To find the entering vector ($\mathbf{a}_k$)

For *predetermined value* of r [obtained from equ. (8·1)], we determine k by using the formula :

$$\frac{\Delta_k}{x_{rk}} = \max_j \left[\frac{\Delta_j}{x_{rj}} , x_{rj} < 0 \right] , \qquad ...(8\cdot2)$$

then the vector $\mathbf{a}_k$ will enter the basis. If this min. is attained for more than one value of k, then degeneracy appears and the problem of cycling can be removed by usual rules for resolution of degeneracy as in simplex method.

In above example, we have

x_2	-1	6/5	0	1	4/5	$-3/5$	0	
x_5	0	0	0	0	1	-1	1	
	$z' = \mathbf{C_B X_B} = -12/5$ $\therefore z = 12/5$		0	0	2/5	1/5	0	$\leftarrow \Delta_j$

Example 3. *Use dual simplex method to solve :*

Min. $z = 3x_1 + x_2$, *subject to* $x_1 + x_2 \geq 1$, $2x_1 + 3x_2 \geq 2$, x_1, *and* $x_2 \geq 0$. **[IAS (Maths) 90]**

Solution. The given problem can be written as

Max. $z' = -3x_1 - x_2$, $z' = -z$, subject to

$$\begin{aligned} -x_1 - x_2 &\leq -1 \\ -2x_1 - 3x_2 &\leq -2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Adding the slack variables x_3 and x_4 to each constraint, respectively, we get

$$-x_1 - x_2 + x_3 \quad = -1$$
$$-2x_1 - 3x_2 \quad + x_4 = -2\,.$$

Writing the constraint equations in matrix form, we have

$$\begin{bmatrix} -1 & -1 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Now the starting dual simplex table can be constructed as *Table 8·5*.

Table 8·5

		$c_j \rightarrow$	-3	-1	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$ (β_1)	$\mathbf{X_4}$ (β_2)	
x_3	0	-1	-1	-1	1	0	
x_4	0	$\rightarrow -2$	-2	$\boxed{-3}$	0	1	
		$z = \mathbf{C_B X_B} = \mathbf{0}$	3	1 $\uparrow$	0	0 $\downarrow$	$\leftarrow \Delta_j$

The solution : $x_1 = 0$, $x_2 = 0$, $x_3 = -1$, $x_4 = -2$ is the starting basic solution which is infeasible but optimal.·We now start with the first iteration of dual simplex method.

First Iteration

To determine the leaving vector (β_r) :

Since $x_{Br} = \min.\ (x_{Bi}\,,\, x_{Bi} < 0) \ = \min\ (x_{B1}\,,\, x_{B2}) = \min\ (-1\,,\, -2) = -2 = x_{B2}$

Hence $r = 2$ .So we must remove the vector β_2 (marked $\downarrow$).

To determine the entering vector a_k for predetermined value of r (= 2) :

Since, $\dfrac{\Delta_k}{x_{2k}} = \max \left[\dfrac{\Delta_1}{x_{21}}, \dfrac{\Delta_2}{x_{22}}, \text{ for } x_{21} < 0, x_{22} < 0\right] \ = \ \max \left[\dfrac{3}{-2}, \dfrac{1}{-3}\right] = \dfrac{-1}{3} = \dfrac{\Delta_2}{x_{22}}$

Hence $k = 2$. So we must enter the vector $\mathbf{a_2}$ corresponding to which $\mathbf{X_2}$ is already given in *Table 8·5*.

To find the transformed table **:**

Here the key element is (-3). In the usual manner, we can get the transformed table as given below :

Table 8·6

		$c_j \rightarrow$	-3	-1	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$ (β_2)	$\mathbf{X_3}$ (β_1)	$\mathbf{X_4}$	
x_3	0	$\rightarrow -1/3$	$-1/3$	0	1	$\boxed{-1/3}$	
x_2	-1	2/3	2/3	1	0	$-1/3$	
		$z' = \mathbf{C_B X_B} = -2/3$ $\therefore z = 2/3$	7/3	0	0 $\downarrow$	1/3 $\uparrow$	$\leftarrow \Delta_j$

$\Delta_1 = \mathbf{C_B X_1} - c_1 = (0\,,\, -1)\,(-\tfrac{1}{3}\,,\, \tfrac{2}{3}) + 3 = 7/3,\ \ \Delta_4 = \mathbf{C_B X_4} - c_4 = (0\,,\, -1)\,(-\tfrac{1}{3}\,,\, -\tfrac{1}{3}) - 0 = \tfrac{1}{3}\,.$

Even now, the corresponding basic solution is infeasible but optimal. So we proceed to second iteration.

Second Iteration

To find the leaving vector (β_r) :

Since $x_{Br} = \min\ [x_{B1}]$, (x_{B2} is ignored because it is not negative). Therefore, $r = 1$.

We we must remove the vector β_1 .

To find the entering vector $\mathbf{a}_k$ ***for predetermined value of*** r **(= 1) :**

Since, $\qquad \dfrac{\Delta_k}{x_{rk}} \equiv \dfrac{\Delta_k}{x_{1k}} = \max. \left[\dfrac{\Delta_1}{x_{11}}, \dfrac{\Delta_4}{x_{14}}\right]$ (because only x_{11} and x_{14} are negative)

$$= \max.\left[\frac{7/3}{-1/3}, \frac{1/3}{-1/3}\right] = \frac{-1}{1} = \frac{\Delta_4}{x_{14}}$$

Hence $k = 4$. So we must enter the vector $\mathbf{a}_4$ corresponding to which **vector $\mathbf{X}_4$** is given in *Table 8·6*.

To find the transformed table :

In *Table* 8·6, the key element is $(-\frac{1}{3})$. In usual manner, we can get the following transformed table.

Table 8·7

		$c_j \rightarrow$	-3	-1	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	
x_4	0	1	1	0	-3	1	
x_2	-1	1	1	1	-1	0	
	$z' = \mathbf{C_B X_B} = -1$ $\therefore z = 1$		2	0	1	0	$\leftarrow \Delta_j$

$\Delta_1 = \mathbf{C_B X_1} - c_1 = (0, -1)(1, 1) + 3 = 2, \ \Delta_3 = \mathbf{C_B X_3} - c_3 = (0, -1)(-3, -1) - 0 = 1.$

At this stage, the solution $x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 1$ becomes feasible and hence it is the required optimal solution with max $z = 1$.

The solution of the dual problem can be obtained from the final table of primal solution as the values of Δ_j corresponding to slack variables of the primal, *i.e.* $w_1 = 1$, $w_2 = 0$.

Also, the dual problem is given as follows :

Min $z_w = w_1 + 2w_2$, *subject to* $w_1 + 2w_2 \leq 3, w_1 + 3w_2 \leq 1$, and $w_1, w_2 \geq 0$.

Example 4. *Use dual simplex method to solve :*

Max. $z = -2x_1 - x_3$, *subject to* $x_1 + x_2 - x_3 \geq 5$, $x_1 - 2x_2 + 4x_3 \geq 8$, *and* $x_1, x_2, x_3 \geq 0$.

[Banasthali (M.Sc.) 93; Roorkee (B.E. IVth) 90]

Solution. The given problem can be written as :

Max. $z = -2x_1 - 0x_2 - x_3$, *subject to* $-x_1 - x_2 + x_3 \leq -5, -x_1 + 2x_2 - 4x_3 \leq -8$, and $x_1, x_2, x_3 \geq 0$.

Adding the slack variables x_4 and x_5 to each constraint, respectively, we get the constraint equations :

$$-x_1 - x_2 + x_3 + x_4 = -5$$
$$-x_1 + 2x_2 - 4x_3 + x_5 = -8.$$

It matrix form, these two equations can be written as :

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & 2 & -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5 \\ -8 \end{bmatrix}.$$

Now, we are able to construct the following starting dual simplex table :

Table 8·8

		$c_j \rightarrow$	-2	0	-1	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$ (β_1)	$\mathbf{X_5}$ (β_2)	
x_4	0	-5	-1	-1	1	1	0	
x_5	0	$\rightarrow -8$	-1	2	$\boxed{-4}$	0	1	
	$z = \mathbf{C_B X_B} = 0$		2	0	1 $\uparrow$	0	0 $\downarrow$	$\leftarrow \Delta_j$

The basic solution, $x_1 = x_2 = x_3 = 0$, $x_4 = -5$, $x_5 = -8$ in *(Table 8·8)*, is infeasible but optimal. So we start with first iteration of dual simplex method.

First Iteration

To determine the leaving vector (β_r).

Since $\qquad x_{Br} = \min.\ [x_{Bi}, x_{Bi} < 0] = \min.\ [x_{B1}, x_{B2}] = \min.\ [-5, -8] = -8 = x_{B2}$,

Hence $r = 2$. So we must remove the vector β_2.

To determine the entering vector $(\mathbf{a}_k)$ ***for predetermined value of*** r $(= 2)$ **:**

Since

$$\frac{\Delta_k}{x_{rk}} \equiv \frac{\Delta_k}{x_{2k}} = \max.\left[\frac{\Delta_1}{x_{21}}, \frac{\Delta_2}{x_{22}}, \frac{\Delta_3}{x_{23}}\right] = \max.\left[\frac{2}{-1}, \ldots, \frac{1}{-4}\right] = \frac{-1}{4} = \frac{\Delta_3}{x_{23}}$$

Therefore, $k = 3$. So we must enter the vector $\mathbf{a}_3$ corresponding to which $\mathbf{X}_3$ is given in *Table* 8·8.

To obtain the transformed table **:**

In *Table* 8·8, -4 is found to be the key element. So, adopting the usual rules of transformation, we get the following transformed table.

Table 8·9

		$c_j \rightarrow$	-2	0	-1	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$ (β_2)	$\mathbf{X_4}$ (β_1)	$\mathbf{X_5}$	
x_4	0	$\rightarrow -7$	$-5/4$	$\boxed{-1/2}$	0	1	$1/4$	
x_3	-1	2	$1/4$	$-1/2$	1	0	$-1/4$	
	$z = \mathbf{C_B X_B} = -2$		$7/4$	$1/2$ $\uparrow$	0	0 $\downarrow$	$1/4$	$\leftarrow \Delta_j$

$$\Delta_1 = \mathbf{C_B X_1} - c_1 = (0, -1)\,(-5/4\,.\,1/4) - 2 = 7/4$$

$$\Delta_2 = \mathbf{C_B X_2} - c_2 = (0, -1)\,(-\tfrac{1}{2}, -\tfrac{1}{2}) - 0 = 1/2$$

$$\Delta_5 = \mathbf{C_B X_5} - c_5 = (0, -1)\,(1/4 - 1/4) - 0 = 1/4\,.$$

The corresponding basic solution is given by $x_1 = x_2 = x_5 = 0,\ x_4 = -7,\ x_3 = 2,\ z = -2$.

Since one infeasibility is removed, the objective function progresses towards the optimal value. We now start with the second iteration.

Second Iteration

To determine the leaving vector (β_r).

Since $\qquad x_{Br} = \min.\ [x_{Bi}, x_{Bi} < 0] = x_{B1} = -7$

Therefore, $r = 1$. So we must remove the vector β_1.

To determine the entering vector $(\mathbf{a}_k)$ ***for predetermined value of*** r $(= 1)$.

Since,

$$\frac{\Delta_k}{x_{rk}} = \frac{\Delta_k}{x_{1k}} = \max.\left[\frac{\Delta_1}{x_{11}}, \frac{\Delta_2}{x_{12}}, \frac{\Delta_3}{x_{13}}\right] = \max.\left[\frac{7/4}{-5/4}, \frac{1/2}{-1/2}, \ldots\right] = \frac{\Delta_2}{x_{12}}$$

Therefore, $k = 2$. So, the entering vector will be $\mathbf{a}_2$ corresponding to which $\mathbf{X}_2$ is given in *Table* 8·9.

To obtain the transformed table **:**

Key element is found to be $(-\frac{1}{2})$. So we get the transformed table as usual.

Table 8·10

		$c_j \rightarrow$	-2	0	-1	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	
x_2	0	14	$5/2$	1	0	-2	$-1/2$	
x_3	-1	9	$3/2$	0	1	-1	$-1/2$	
	$z = -9$		$1/2$	0	0	1	$1/2$	$\leftarrow \Delta_j$

$$\Delta_1 = \mathbf{C_B X_1} - c_1 = (0, -1)\,(5/2, 3/2) - 2 = \tfrac{1}{2}$$

$$\Delta_4 = \mathbf{C_B X_4} - c_4 = (0\,,-1)\,(-2\,,-1) - 0 = 1$$
$$\Delta_5 = \mathbf{C_B X_5} - c_5 = (0\,,-1)\,(-\tfrac{1}{2}\,,-\tfrac{1}{2}) - 0 = \tfrac{1}{2}\,.$$

The associated basic solution is given by $x_1 = 0\,, x_2 = 14\,, x_3 = 9\,, z = -9$.

At this iteration, all the basic variables are non-negative, and consequently the calculations terminate with the above mentioned optimal solution.

Also, the optimal values of the dual variables $w_1\,, w_2$, as read from the final *Table* 8·10 , are given by

$$w_1 = \Delta_4 = 1\,,\ w_2 = \Delta_5 = \tfrac{1}{2}\,,$$

where the dual problem is given as follows :

Min. $z_w = 5w_1 + 8w_2$, subject to $w_1 + w_2 \le -2, w_1 - 2w_2 \le 0, -w_1 - 4w_2 \le -1,$ *and* $w_1\,, w_2 \ge 0$.

Example 5. *Use dual simplex method to solve the following L.P.P.* :

Min. $\quad z = 6x_1 + 7x_2 + 3x_3 + 5x_4$, *subject to*

$5x_1 + 6x_2 - 3x_3 + 4x_4 \ge 12, x_2 + 5x_3 - 6x_4 \ge 10, 2x_1 + 5x_2 + x_3 + x_4 \ge 8,$ *and* $x_1\,, x_2\,, x_3\,, x_4 \ge 0$

[Meerut 91, 90]

Solution. ***Step* 1.** The given L.P.P.is within in standard primal form as follows :

$$\textit{Max.}\quad z' = -6x_1 - 7x_2 - 3x_3 - 5x_4,\ z' = -z$$
$$\begin{aligned} \textit{s.t.}\quad -5x_1 - 6x_2 + 3x_3 - 4x_4 &\le -12 \\ -x_2 - 5x_3 + 6x_4 &\le -10 \\ -2x_1 - 5x_2 - x_3 - x_4 &\le -8 \\ \textit{and}\qquad x_1\,, x_2\,, x_3\,, x_4 &\ge 0\,. \end{aligned}$$

Since objective function is of maximization and all $c_j < 0$, we can solve this L.P.P. by dual simplex algorithm.

***Step* 2.** Introducing the slack variables $x_5\,, x_6$ and x_7 the constraints of the above problem reduce to the following equalities :

$$\begin{aligned} -5x_1 - 6x_2 + 3x_3 - 4x_4 + x_5 \qquad\qquad &= -12 \\ -x_2 - 5x_3 + 6x_4 \qquad + x_6 \qquad &= -10 \\ -2x_1 - 5x_2 - x_3 - x_4 \qquad\qquad + x_7 &= -8 \end{aligned}$$

$\therefore$ The starting basic solution to the primal is $x_1 = x_2 = x_3 = x_4 = 0, x_5 = -12\,, x_6 = -10\,, x_7 = -8$, which is infeasible.

The starting simplex table is as follows.

Table 8·11

	$c_j \rightarrow$		-6	-7	-3	-5	0	0	0
Basic Var.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$ β_1	$\mathbf{X_6}$ β_2	$\mathbf{X_7}$ β_3
x_5	0 $\rightarrow$	-12	-5	$\boxed{-6}$	3	-4	1	0	0$\rightarrow$
x_6	0	-10	0	-1	-5	6	0	1	0
x_7	0	-8	-2	-5	-1	-1	0	0	1
	$z' = 0$		6	7$\uparrow$	3	5	0$\downarrow$	0	0$\leftarrow \Delta_j$

$$\Delta_1 = \mathbf{C_B X_1} - c_1 = 6\,, \Delta_2 = 7, \Delta_3 = 3\,, \Delta_4 = 5\,, \Delta_5 = 0 = \Delta_6 = \Delta_7\,.$$

Thus the starting basic solution is infeasible but optimal.

To determine the leaving vector (β_r) :

Since $\qquad x_{Br} = \text{Min}\,(\mathbf{x_{Bi}}\,, x_{Bi} < 0) = \text{Min}\,(-12\,, -10\,, -8) = -12 = x_{B1}$

$\therefore\quad r = 1$, *i.e.*, β_1 $(= \mathbf{X_5})$ is the leaving vector.

To determine the entering vector $(\mathbf{a_k})$ ***for predetermined value of*** r $(= 1)$:

$$\frac{\Delta_k}{x_{rk}} = \frac{\Delta_k}{x_{1k}} = \underset{j}{\text{Max}} \left\{ \frac{\Delta_j}{x_{1j}}\,, x_{1j} < 0 \right\} = \text{Max} \left\{ \frac{\Delta_1}{x_{11}}\,, \frac{\Delta_2}{x_{12}}\,, \frac{\Delta_4}{x_{14}} \right\}$$

$$= \text{Max}\left\{\frac{6}{-5}, \frac{7}{-6}, \frac{5}{-4}\right\} = \frac{-7}{6} = \frac{\Delta_2}{x_{12}}.$$

Therefore, $k = 2$, *i.e.*, $\mathbf{a}_2$ $(= \mathbf{X}_2)$ is the entering vector. Hence key element $= x_{12} = -6$.

Proceeding as usual, the second simplex table is as follows.

Table 8·12

		$c_j \rightarrow$	-6	-7	-3	-5	0	0	0	
Basic Var.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{X_6}$	$\mathbf{X_7}$	
x_2	-7	2	$5/6$	1	$-1/2$	$2/3$	$-1/6$	0	0	
x_6	0	$\rightarrow \boxed{-8}$	$5/6$	0	$\boxed{-11/2}$	$20/3$	$-1/6$	1	$0 \rightarrow$	
x_7	0	2	$13/6$	0	$-7/2$	$7/3$	$-5/6$	0	1	
	$z' = -14$		$1/6$	0	$13/2$ $\uparrow$	$1/3$	$7/6$	0 $\downarrow$	0	$\leftarrow \Delta_j$

The solution given in Table (8·12) is : $x_1 = x_3 = x_4 = x_5 = 0$, $x_2 = 2$, $x_6 = -8$, $x_7 = 2$, which is infeasible but optimal. Therefore, it can be improved further.

To determine the leaving vector (β_r) :

Since $\quad x_{Br} = \text{Min}\,(x_{Bi}, x_{Bi} \le 0) = \text{Min}\,(x_{B2}) = \text{Min}\,(-8) = -8 = x_{B2}$

Therefore, $r = 2$, *i.e.* β_2 $(= \mathbf{X}_6)$ is the leaving vector.

To determine the entering vector $(\mathbf{a}_k)$ ***for predetermined value of*** r **(= 2)**

$$\frac{\Delta_k}{x_{rk}} = \frac{\Delta_k}{x_{2k}} = \underset{j}{\text{Max}}\left\{\frac{\Delta_j}{x_{2j}}, x_{2j} < 0\right\} = \text{Max}\left\{\frac{\Delta_3}{x_{23}}, \frac{\Delta_5}{x_{25}}\right\}$$

$$= \text{Max}\left\{\frac{13/2}{-11/2}, \frac{7/6}{-1/6}\right\} = \text{Max}\left\{\frac{-13}{11}, \frac{-7}{1}\right\} = \frac{-13}{11} = \frac{\Delta_3}{x_{23}}$$

Therefore, $k = 3$, *i.e.* $\mathbf{a}_3$ $(= \mathbf{X}_3)$ is the entering vector. Hence key element $= x_{23} = -11/2$.

Proceeding as usual, the ***third simplex table*** is obtained as follows :

Table 8·13

		$c_j \rightarrow$	-6	-7	-3	-5	0	0	0	
Basic Var.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{X_6}$	$\mathbf{X_7}$	
x_2	-7	$30/11$	$25/33$	1	0	$2/33$	$-5/33$	$-1/11$	0	
x_3	-3	$16/11$	$-5/33$	0	1	$-40/33$	$1/33$	$-2/11$	0	
x_7	0	$78/11$	$18/11$	0	0	$-21/11$	$-8/11$	$-7/11$	1	
	$z' = -258/11$		$38/33$	0	0	$271/33$	$32/33$	$13/11$	0	Δ_j

The solution given in *Table 8·13* is : $x_1 = x_4 = x_5 = x_6 = 0$, $x_2 = 30/11$, $x_3 = 16/11$ and $x_7 = 78/11$, which is feasible and optimal.

Hence the optimal feasible solution of the given L.P.P. is :

$$x_1 = 0,\ x_2 = 30/11,\ x_3 = 16/11,\ x_4 = 0 \text{ and Min. } z = -\text{Max. } z' = 258/11.$$

Remark. It is interesting to note in the dual simplex method that we seek to maintain dual feasibility but remove the primal infeasibilities. The starting basic solution in this method is obviously *dual feasible.*

8.4. ADVANTAGE OF DUAL SIMPLEX METHOD OVER SIMPLEX METHOD

The main advantage of dual simplex method over the usual simplex method is that we do not require any artificial variables in the dual simplex method. Hence a lot of labour is saved whenever this method is applicable.

8.5. DIFFERENCE BETWEEN SIMPLEX AND DUAL SIMPLEX METHODS

The dual simplex method is similar to the standard simplex method except that in the latter the starting initial basic solution is feasible but not optimum while in the former it is infeasible but optimum or better than optimum. The dual simplex method works towards feasibility while simplex method works towards optimality.

8.6. SUMMARY AND COMPUTER APPLICATIONS

The iterative procedure for dual simplex algorithm may be summarized as follows :

Step 1. First convert the minimization LPP into that of maximization, if it is given in the minimization form.

Step 2. Convert the ' $\geq$ ' type inequalities of given LPP, if any, into those of ' $\leq$ ' type by multiplying the corresponding constraints by -1.

Step 3. Introduce slack variables in the constraints of the given problem and obtain an initial basic solution. Put this solution in the starting dual simplex table.

Step 4. Test the nature of $z_j - c_j$ in the starting table.

(*i*) If all $z_j - c_j$ and x_{Bi} are non-negative for all i and j, then an optimum basic feasible solution has been attained.

(*ii*) If all $z_j - c_j$ are non-negative and at least one basic variable, x_{Bi}, is negative, then go to ***step* 5.**

(*iii*) If at least one $z_j - c_j$ is negative, the method is not applicable to the given problem.

Step 5. Select the most negative x_{Bi}. The corresponding basis vector then leaves the basis set **B**. Let x_{Br} be the most negative basic variable so that β_r leaves the basis set **B**.

Step 6. Test the nature of x_{rj} , $j = 1, 2, \ldots, n$.

(*i*) If all x_{rj} are non-negative, there does not exist any feasible solution to the given problem.

(*ii*) If at least one x_{rj} is negative, compute the replacement ratios

$$\left\{ \frac{z_j - c_j}{x_{rj}}, x_{rj} < 0 \right\}, j = 1, 2, \ldots, n,$$

and choose the maximum of these. The column vector corresponding to x_{rk} then enters the basis set **B**.

Step 7. Test the new iterated dual simplex table for optimality.

Repeat the entire procedure until either an optimum feasible solution has been attained in a finite number of steps or there is an indication of the non-existence of a feasible solution.

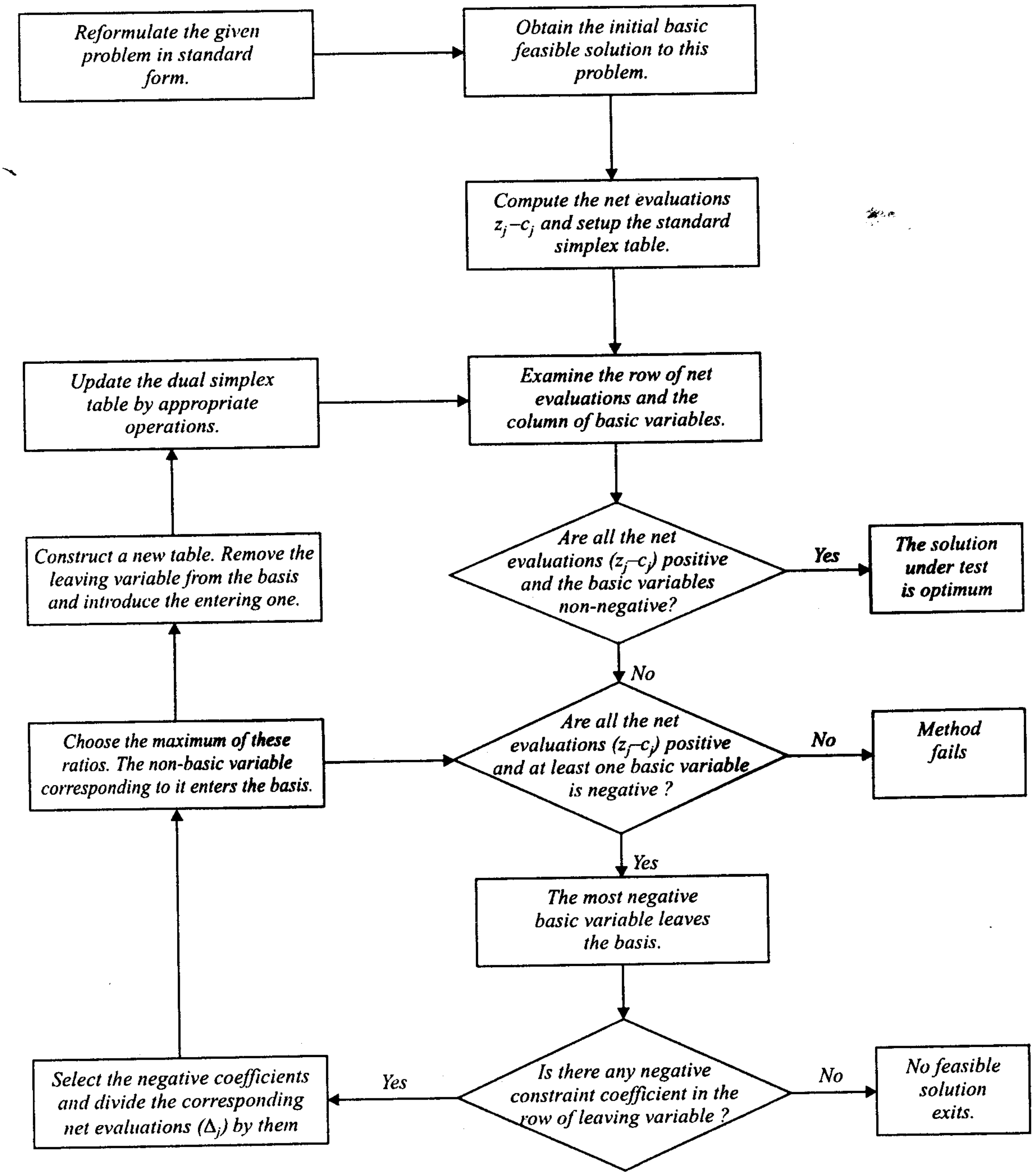
For computer applications, flow-chart of dual simplex method is given as follows :

Q. 1. What is the essential difference between regular simplex method and dual simplex method ? **[Meerut M.Sc. (Meth) 90]**

2. Give the outlines of dual simplex method. **[Meerut M.Sc. (Math.) 90]**

3. What is dual simplex algorithm ? State various steps involved in the dual simplex algorithm.

FLOWCHART OF DUAL SIMPLEX METHOD



EXAMINATION PROBLEMS

Use dual simplex method to solve the following linear programming problems :

1. Max. $z = -3x_1 - 2x_2$
subject to
$x_1 + x_2 \geq 1$
$x_1 + x_2 \leq 7$
$x_1 + 2x_2 \leq 10$
$x_2 \leq 3.$

[Ans. $x_1 = 0$, $x_2 = 1$, Max. $z = -2$]

2. Max. $z = -2x_1 - 2x_2 - 4x_3$
subject to
$2x_1 + 3x_2 + 5x_3 \geq 2$
$3x_1 + x_2 + 7x_3 \leq 3$
$x_1 , x_2 , x_3 \geq 0.$

[Ans. $x_1 = 0$, $x_2 = 2/3$, $x_3 = 0$, $z^* = 4/3$]

3. Min. $x_1 + 2x_2 + 3x_3$,
subject to
$2x_1 - x_2 + x_3 \geq 4$
$x_1 + x_2 + 2x_3 \geq 8$
$x_2 - x_3 \geq 2$
$x_1 , x_2 , x_3 \geq 0$. **[AIMS(Banglore)02]**

[Hint .Start with a basic-nonfeasible solution.]
[Ans. $x_1 = 6$, $x_2 = 2$, $x_3 = 0$, min. $z = 10$]

4. Min. $z = 6x_1 + x_2$,
subject to
$2x_1 + x_2 \geq 3$
$x_1 - x_2 \geq 0$
$x_1 , x_2 \geq 0.$

[Ans. $x_1 = 1$, $x_2 = 1$, min. $z = 7$]

5. Min. $z = x_1 + x_2$,
subject to
$2x_1 + x_2 \geq 2$
$-x_1 - x_2 \geq 1$
$x_1 , x_2 \geq 0.$

[Ans. Pseudo optimum basic feasible solution.]

6. Min. $z = 10x_1 + 6x_2 + 2x_3$,
subject to
$-x_1 + x_2 + x_3 \geq 1$
$3x_1 + x_2 - x_3 \geq 2$
$x_1 , x_2 , x_3 \geq 0.$ **[VTU (BE com.) 02]**

[Ans. $x_1 = 1/4$, $x_2 = 5/4$, $x_3 = 0$, min. $z = 10$]

7. Min. $z = 80x_1 + 60x_2 + 80x_3$,
subject to
$x_2 + 2x_2 + 3x_3 \geq 4$
$2x_1 \quad + 3x_3 \geq 3$
$2x_1 + 2x_2 + x_3 \geq 4$
$4x_1 \quad + x_2 + x_3 \geq 6$
$x_1 , x_2 , x_3 \geq 0$

[Ans. $x_1 = 16/13$, $x_2 = 6/13$, $x_3 = 3/13$, min. $z = 2280/13$]

8. Min. $z = 3x_1 + 2x_2 + x_3 + 4x_4$,
subject to
$2x_1 + 4x_2 + 5x_3 + x_4 \geq 10$
$3x_1 - x_2 + 7x_3 - 2x_4 \geq 2$
$5x_1 + 2x_2 + x_3 + 6x_4 \geq 15$
$x_1 , x_2 , x_3 , x_4 \geq 0$.

[Ans. $x_1 = 65/23$, $x_2 = 0$, $x_3 = 20/23$, $x_4 = 0$, min. $z = 215/23$]

9. Min. $z = x_1 + 2x_2$
subject to
$2x_1 + x_2 \geq 4$
$x_1 + 2x_2 \geq 7$
$x_1 , x_2 \geq 0$.

[Ans. $x_1 = 0$, $x_2 = 2$, min. $z = 4$]

10. Maximize $z = -4x_1 - 6x_2 - 18x_3$
subject to $x_1 + 3x_3 \geq 3$
$x_2 + 2x_3 \geq 5$
$x_1, x_2, x_3 \geq 0$

[Ans. $x_1 = 0$, $x_2 = 3$, $x_3 = 1$, max. $z = -36$.]

11. Minimize $z = 2x_1 + 2x_2$
subject to $x_1 + 2x_2 \geq 1$
$2x_1 + x_2 \geq 1$
$x_1 \geq 0$ and $x_2 \geq 0$

[Ans. $x_1 = 1/3$, $x_2 = 1/3$, min $z = 4/3$]

12. Use dual simplex method to obtain zeroth and first iteration for the problem :
$-2x_1 - x_2 + 5x_3 \geq 2$, $3x_1 + 2x_2 + 4x_3 \geq 16$, $3x_1 + 5x_2 + 4x_3 = z$ (min.), and $x_1 , x_2 , x_3 \geq 0$.
Write complementary basis corresponding to first iterate. Write the simplex multipliers with respect to the basis of first iterate. Verify these results.
[Ans. $x_1 = x_2 = 0$, $x_3 = 4$, min $z = 16$]

13. (a) Show that the value of the objective function of the dual for any feasible solution is never less than the value of the objective function of the primal corresponding to any feasible solution.
(b) Write the dual corresponding to $x + y + 2z \leq 120$, $3x - 2y - z \geq 90$, $2x + 4y + 2z = 10$,, $5x + 8y + 10z = R$ (max.) $x , y , z \geq 0$.Use dual simplex or simplex method and obtain zeroth and first iterates of the dual. Write the simplex multipliers corresponding to the basis of the first iterate.

14. Show with the help of an example how when one solves an LP problem by simplex method going through infeasible but better than optimal solution, one indirectly goes through infeasible but better than optimal solution of the dual LP problem. How this fact is utilized in the solution of the dual.

15. What is the essential difference between regular simplex and dual simplex method ? **[Meerut (LP) 90]**

16. Find optimum solution of the following problem by not using artificial variables :
Min. $z = 10x_1 + 10x_2$, s.t. $x_1 + x_2 \geq 10$, $3x_1 + 2x_2 \geq 24$, $x_1 \geq 0$, $x_2 \geq 0$, **[Delhi (MCI) 2000]**

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9

SENSITIVITY (POST-OPTIMALITY) ANALYSIS

9.1. INTRODUCTION

So far, we have assumed that all the coefficients of a linear programming problem are prescribed. But, the optimal solution of the linear programming problem (max. $z = \mathbf{CX}$, subject to $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \geq \mathbf{0}$) depends upon the parameters (c_j, a_{ij} and b_i) of the problem. The parameters of the problem are usually not known with complete certainty, *i.e.*, the a_{ij}, b_i and c_j are estimates or they vary over time. For example, in a diet problem, the cost of any particular feed will vary from time to time. If the optimal value of the objective function is relatively sensitive to changes in certain parameters, special care should be taken in estimating these parameters and in selecting a solution which does well for most of their likely values. Then, it is quite important to know the range of cost for which the solution remains optimal. Therefore, it seems desirable to see as to how sensitive the optimal solution is with regard to discrete changes in parameters of the problem.

Thus, the investigation that deals with changes in the optimal solution due to changes in the parameters (a_{ij}, b_i and c_j) is called ***sensitivity analysis*** or ***post-optimality analysis.***

The objective of sensitivity analysis is to reduce the additional computational effort considerably which arise in solving the problem a new. The changes in the linear programming problem which are usually studied by sensitivity analysis include :

1. **Coefficients (c_j) of the objective function.** These include :
   (a) Coefficients of basic variables ($c_j \in \mathbf{C_B}$)   (b) Coefficients of non-basic variables ($c_j \notin \mathbf{C_B}$)
2. **Change in the right-hand side constants (b_i).**
3. **Changes in a_{ij} (the components of matrix A).** These include :
   (a) Coefficients of the basic variables ($a_{ij} \in \mathbf{B}$)   (b) Coefficients of the non-basic variables ($a_{ij} \notin \mathbf{B}$)
4. **Addition of new variables to the problem.**
5. **Addition of new (or secondary) constraint(s).**

In general, these changes may result in one of the following three cases :

Case I. The optimal solution remains unchanged, that is, the basic variables and their values remain essentially unchanged.

Case II. The *basic* variables remain the same, but their values are changed.

Case III. The basic solution changes completely.

Before introducing the techniques for testing the different changes in the linear programming problem, we shall first prove an *important lemma* in the folloiwng section.

9.2. AN IMPORTANT LEMMA

Lemma. If $d_j \geq 0$, $d_j + \lambda d_j' \geq 0$, $j = 1, 2, \ldots, n$, then

$$\underset{(d_j' > 0)}{\underset{j}{\text{Max.}}} \left[-\frac{d_j}{d_j'}\right] \leq \lambda \leq \underset{(d_j' < 0)}{\underset{j}{\text{Min.}}} \left[-\frac{d_j}{d_j'}\right].$$

Proof. Since $d_j \geq 0$ and $d_j + \lambda d_j' \geq 0$ $(j = 1, 2, \ldots, n)$, we have the following *three* possibilities :

(i) For those j for which $d_j' > 0$, we have

$$\lambda \geq -\frac{d_j}{d_j'} \quad \text{or} \quad \lambda \geq \underset{j}{\text{Max}}\left[\frac{-d_j}{d_j'}\right], (d_j' > 0)$$

(ii) For those j for which $d_j' < 0$, we have

$$\lambda \leq -\frac{d_j}{d_j'} \quad \text{or} \quad \lambda \leq \underset{j}{\text{Min}}\left[\frac{-d_j}{d_j'}\right], (d_j' < 0)$$

(iii) For those j for which $d_j' = 0$, we have λ as unrestricted.

Thus, from (i) and (ii) above, we conclude that λ must satisfy the relationship :

$$\boxed{\underset{\substack{j \\ (d_j' > 0)}}{\text{Max}}\left[\frac{-d_j}{d_j'}\right] \leq \lambda \leq \underset{\substack{j \\ (d_j' < 0)}}{\text{Min.}}\left[\frac{-d_j}{d_j'}\right].} \qquad \ldots(9{\cdot}1)$$

Also, if $d_j' \geq 0$ for all j, then there is no upper bound. Similarly, if $d_j' \leq 0$ for all j, then there is no lower bound.

This completes the proof of the *lemma*.

9.3. CHANGES IN THE COEFFICIENTS (c_j) OF THE OBJECTIVE FUNCTION

Let us consider the linear programming problem

$$\text{Max.}\, z = \mathbf{CX}\,, \text{subject to } \mathbf{AX} = \mathbf{b}\,, \mathbf{X} \geq \mathbf{0}\,, \qquad \ldots(9{\cdot}2)$$

for which $\mathbf{X_B}$ is the optimal basic feasible solution and $\mathbf{B}$ is an optimal basis matrix. We assume that our problem is non-degenerate and that we have had a basic feasible solution to (9·2).

Then, we have

$$\mathbf{X_B} = \mathbf{B}^{-1}\,\mathbf{b}\,. \qquad \ldots(9{\cdot}3)$$

Obviously, (9·3) shows that if we change some component c_j of vector $\mathbf{C}$, then $\mathbf{X_B}$ will not change ($\mathbf{X_B}$ is independent of $\mathbf{C}$). Also, $\mathbf{x_B}$ will always remain basic feasible solution. But, however, the optimality conditions

$$\Delta_j \equiv z_j - c_j \geq 0 \text{ for all } j\,,$$

which are satisfied for optimal solution $\mathbf{X_B}$ before changing c_j, will not (necessarily) be satisfied when c_j is changed to $\hat{c}_j$, *i.e.* the inequality $z_j - c_j \geq 0$ may change if c_j is changed. Furthermore, the variations in the vector $\mathbf{C}$ can be made in two possible ways :

(i) Only that component c_j of vector $\mathbf{C}$ is changed which is the coefficient of non-basic variable x_j in the objective function, *i.e.*, $c_j \notin \mathbf{C_B}$.

(ii) Only that component c_j of vector $\mathbf{C}$ is changed which is the coefficient of basic variable x_j in the objective function, *i.e.* $c_j \in \mathbf{C_B}$.

Case I. Variation in $c_j \notin \mathbf{C_B}$:

Let us assume that $c_j \notin \mathbf{C_B}$ is changed to $(c_j + \Delta c_j)$. Since $\mathbf{X_B}$ is an optimal solution, therefore $z_j - c_j \geq 0$, for all j.

If with this change the optimality of $\mathbf{X_B}$ is to be preserved, then Δc_j must satisfy the optimality condition

$$z_j - (c_j + \Delta c_j) \geq 0\,. \qquad \ldots(9{\cdot}4)$$

Since $z_j = \mathbf{C_B}\,\mathbf{X_j}$ does not depend upon $c_j \notin \mathbf{C_B}$, we shall have

$$\Delta c_j \leq z_j - c_j \quad \text{or} \quad \Delta c_j \leq \Delta_j \qquad \ldots(9{\cdot}5)$$

Thus, if $c_j \notin \mathbf{C_B}$ is replaced by $(c_j + \Delta c_j)$ such that $\Delta c_j \leq z_j - c_j$; then the optimal value of the objective function and the optimal solution as well will not be affected.

Case II. Variations in $c_j \in \mathbf{C_B}$ (say $c_j = c_{Br}$) :

We have

$$z_j = \mathbf{C_B}\,\mathbf{X_j} = \sum_{i=1}^{m} c_{Bi}\, x_{ij}\,.$$

Therefore, if we change c_{Br} to $c_{Br} + \Delta c_{Br}$, then the value of z_j is affected and is given by

$$\hat{z}_j = \sum_{i=1, i \neq r}^{m} c_{Bi}\, x_{ij} + (c_{Br} + \Delta c_{Br})\, x_{rj}\, .$$

Also, with this change $\hat{z}_j - c_j = \left[\sum_{i=1, i \neq r}^{m} c_{Bi}\, x_{ij} + (c_{Br} + \Delta c_{Br})\, x_{rj} \right] - c_j$

$$= \left[\sum_{i=1}^{m} c_{Bi}\, x_{ij} + x_{rj}\, \Delta c_{Br} \right] - c_j$$

$$= [z_j + x_{rj}\, \Delta c_{Br}] - c_j = (z_j - c_j) + x_{rj}\, \Delta c_{Br}\, .$$

If the solution $\mathbf{X_B}$ remains optimal after such variation, then Δc_{Br} must satisfy the optimality condition

$$\hat{z}_j - c_j \geq 0 \quad \text{or} \quad (z_j - c_j) + x_{rj}\, \Delta c_{Br} \geq 0. \qquad ...(9{\cdot}6)$$

Since $z_j - c_j \geq 0$ (because $\mathbf{X_B}$ is optimal initially), $(z_j - c_j) + \Delta c_{Br}\, x_{rj} \geq 0$ [from (9·6)], the *lemma* (9·1) can be applied by replacing, $d_j \to z_j - c_j$, $d_j' \to x_{rj}$ and $\lambda \to \Delta c_{Br}$,

and we obtain
$$\boxed{\underset{x_{rj} > 0}{\underset{j}{\textbf{Max.}}} \left[\frac{-\Delta_j}{x_{rj}} \right] \leq \Delta c_{Br} \leq \underset{x_{rj} < 0}{\underset{j}{\textbf{Min.}}} \left[\frac{-\Delta_j}{x_{rj}} \right].} \qquad ...(9{\cdot}7)$$

Furthermore, the value of objective function may change if c_{Br} is changed to $c_{Br} + \Delta c_{Br}$ satisfying (9·7). To determine how much value of objective function will change, we have

$$z = \mathbf{C_B X_B} = \sum_{i=1}^{m} c_{Bi}\, x_{Bi}\, . \qquad ...(9{\cdot}8)$$

But, c_{Br} is replaced by $c_{Br} + \Delta c_{Br}$, therefore the new value of the objective function now becomes :

$$\hat{z} = \sum_{i=1, i \neq r}^{m} c_{Bi}\, x_{Bi} + (c_{Br} + \Delta c_{Br})\, x_{Br} = \left(\sum_{i=1, i \neq r}^{m} c_{Bi}\, x_{Bi} + c_{Br}\, x_{Br} \right) + \Delta c_{Br}\, x_{Br} = \sum_{i=1}^{m} c_{Bi}\, x_{Bi} + \Delta c_{Br}\, x_{Br}$$

or
$$\hat{z} = z + \Delta c_{Br}\, x_{Br}\, . \qquad ...(9{\cdot}9)$$

Thus, if Δc_{Br} satisfies the relationship (9·7), the solution $\mathbf{X_B}$ will remain optimal but the value of objective function will be further improved by an amount $\Delta c_{Br}\, x_{Br}$ [from 9·9].

The following numerical examples will make the above discussion clear.

9.4. ILLUSTRATIVE EXAMPLES

Example 1. ***The following table gives the optimal solution of a linear programming problem [Example 5, page 123]***

Table 9·1

		$c_j \to$	3	5	4	0	0	0	
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{X_6}$	
x_2	5	50/41	0	1	0	15/41	8/41	− 10/41	
x_3	4	62/41	0	0	1	− 6/41	5/41	4/41	
x_1	3	89/41	1	0	0	− 2/41	− 12/41	15/41	
	$z = \mathbf{C_B X_B} = 765/41$		0	0	0	45/41	24/41	11/41	$\leftarrow \Delta_j$

How much c_3 and c_4 can be increased before the present basic solution will no longer be optimal. Also, find the change in the value of the objective function, if possible.

Solution.

Case I. (Variation in c_4). Since $\mathbf{C_B} = [5, 4, 3] = [c_2, c_3, c_1]$, therefore c_4 does not belong to $\mathbf{C_B}$. We have to find the range under which c_4 can vary so that the solution remains optimal.

Therefore, from (9·5), we have

$$\Delta c_4 \le z_4 - c_4 \quad \text{or} \quad \Delta c_4 \le \Delta_4, \quad i.e. \quad \Delta c_4 \le (45/41).$$

Thus, the range over which c_4 can vary, maintaining the optimality of the solution, is given by

$$-\infty < c_4 \le c_4 + \Delta c_4 \quad \text{or} \quad -\infty < c_4 \le 0 + 45/41 \text{ (because } c_4 = 0) \quad \text{or} \quad -\infty < c_4 \le 45/41.$$

Also, the value of the objective function will remain the same, *i.e.* $z = 765/41$.

Case II. (Variation in c_3). Since $(c_{B1}, c_{B2}, c_{B3}) = (5, 4, 3) = (c_2, c_3, c_1)$, therefore $c_3 = c_{B2}$. So we have to find limits of Δc_{B2} when c_{B2} is changed to $(c_{B2} + \Delta c_{B2})$ maintaining the optimality of the solution.

From (9·7), we have

$$\underset{x_{2j}>0}{\underset{j}{\text{Max.}}} \left[\frac{-\Delta_j}{x_{2j}}\right] \le \Delta c_{B2} \le \underset{x_{2j}<0}{\underset{j}{\text{Min}}} \left[\frac{-\Delta_j}{x_{2j}}\right], \; \Delta_j = z_j - c_j$$

or

$$\underset{x_{2j}>0}{\underset{j}{\text{Max.}}} \left[\frac{-\Delta_3}{x_{23}}, \frac{-\Delta_5}{x_{25}}, \frac{-\Delta_6}{x_{26}}\right] \le \Delta c_{B2} \le \underset{x_{2j}<0}{\underset{j}{Min.}} \left[\frac{-\Delta_4}{x_{24}}\right]$$

or

$$\underset{x_{2j}>0}{\underset{j}{\text{Max.}}} \left[\frac{0}{1}, \frac{-24/41}{5/41}, \frac{-11/41}{4/41}\right] \le \Delta c_{B2} \le \underset{x_{2j}<0}{\underset{j}{\text{Min.}}} \left[\frac{-45/41}{-6/41}\right]$$

or

$$0 \le \Delta c_{B2} \le 45/6.$$

Therefore, $c_{B2} - 0 \le c_{B2} \le c_{B2} + 45/6$ or $4 - 0 \le c_{B2} \le 4 + 45/6$ or $4 \le c_{B2} \le 23/2$ or $4 \le c_3 \le 23/2$.

To compute changes in the value of objective function.

Since $z = 765/41$, $0 \le \Delta c_{B2} \le 45/6$, $x_{B2} = 62/41$, and using the result (9·9), we can find the new value

$$\hat{z} = \frac{765}{41} + \left(0 \le \Delta c_{B2} \le \frac{45}{6}\right) \times \frac{62}{41}.$$

Example 2. *Find the optimum solution to the LP problem : Maximize* $z = 15x_1 + 45x_2$ *subject to the constraints :* $x_1 + 16x_2 \le 240$, $5x_1 + 2x_2 \le 162$, $x_2 \le 50$, *and* $x_1, x_2 \ge 0$.

If maximum $z = \Sigma c_j x_j, j = 1, 2,$ *and* c_2 *is kept fixed at 45, determine how much* c_1 *can be changed without affecting the above solution.*

Solution. Introducing the slack variables $x_3 \ge 0, x_4 \ge 0, x_5 \ge 0$ in the constraints of the given LP problem and then solving the resulting problem by simplex method, the following optimum simplex table is obtained.

Table 9·2. Optimum Table

		$c_j \to$	15	45	0	0	0
BASIC VARI.	C_B	X_B	X_1	X_2	X_3	X_4	X_5
x_2	45	173/13	0	1	5/78	1/78	0
x_1	15	352/13	1	0	− 1/39	8/39	0
x_5	0	477/13	0	0	− 5/78	1/78	1
NONBASIC $x_3 = x_4 = 0$	$z = 1005$		0	0	5/2	5/2	0

It is observed that X_2, X_1 and X_5 form the basis matrix. Here $(c_{B1}, c_{B2}, c_{B3}) = (45, 15, 0) = (c_2, c_1, c_5)$

Since c_2 is kept fixed and c_1 can vary, therefore an optimum basic feasible solution will maintain its optimality if the change Δc_1 in c_1 satisfies the relationships :

$$\underset{x_{2j}>0}{max.} \left[\frac{-\Delta_j}{x_{2j}}\right] \le \Delta c_{B2} \le \underset{x_{2j}<0}{\text{min}} \left[\frac{-\Delta_j}{x_{2j}}\right] \qquad (\because \; c_1 = c_{B2})$$

or

$$\max\left[\frac{0}{1}, \frac{-5/2}{8/39}\right] \le \Delta c_1 \le \left[\frac{-5/2}{-1/39}\right] \quad \text{or} \quad 0 \le \Delta c_1 \le 195/2.$$

Example 3. *Given the linear programming problem : Maximize* $z = 3x_1 + 5x_2$, *subject to the constraints :*

$$3x_1 + 2x_2 \le 18, \quad x_1 \le 4, \quad x_2 \le 6, \; and \; x_1, x_2 \ge 0.$$

(i) *Determine an optimum solution to the LP problem.*

(ii) *Discuss the effect on the optimality of the solution when the objective function is changed to* $z = 3x_1 + x_2$.

Solution. Introducing the slack variables $x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$, and then solving the reformulated LP problem by usual simplex method, the optimum simplex table is obtained as below.

Table 9·3. Optimum Table

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	
x_1	3	2	1	0	1/3	0	−2/3	
x_4	0	0	0	0	−2/3	1	4/3	
x_2	5	6	0	1	0	0	1	
NONBASIC VAR. $x_3 = x_5 = 0$.		$z = 36$	0	0	1	0	3	$\leftarrow \Delta_j$

It is observed that $\mathbf{X}_1, \mathbf{X}_4, \mathbf{X}_2$ are the basis vectors in above optimum table, and

$$\mathbf{B}^{-1} = [\mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5] = \begin{bmatrix} 1/3 & 0 & -2/3 \\ -2/3 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the objective function is now changed to $z = 3x_1 + x_2$, new $\mathbf{C_B}$ becomes (3, 0, 1) [in place of (3, 0, 5)].

Then,
$$\mathbf{C_B B}^{-1} = (3, 0, 1) \begin{bmatrix} 1/3 & 0 & -2/3 \\ -2/3 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} = (1, 0, -1).$$

Hence the new net-evaluations for $\mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5$ become

$$\Delta_3 = \mathbf{C_B X}_3 - c_3 = (3, 0, 1)\left(\frac{1}{3}, \frac{-2}{3}, 0\right) = 1,\ \Delta_4 = \mathbf{C_B X}_4 - c_4 = (3, 0, 1)\ (0, 1, 0) = 0$$

$$\Delta_5 = \mathbf{C_B X}_5 - c_5 = (3, 0, 1)\left(-\frac{2}{3}, \frac{4}{3}, 1\right) = -1$$

This shows that $\mathbf{X}_5$ enters the basis in the next iteration. Now apply simplex method to obtain the optimum solution.

The new simplex table becomes as *Table* 9·4.

Table 9·4

		$c_j \rightarrow$	3	1	0	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	MIN RATIO (X_B/X_5)
x_1	3	2	1	0	1/3	0	−2/3	—
x_4	0	0	0	0	−2/3	1	$\boxed{4/3}$	0 ←
x_2	1	6	0	1	0	0	1	6
NON-BASIC $x_3 = x_5 = 0$		$z = 12$	0	0	1	0 ↓	−1 ↑	$\leftarrow \Delta_j$
x_1	3	2	1	0	0	1/2	0	
x_5	0	0	0	0	−1/2	3/4	1	
x_2	1	6	0	1	1/2	−3/4	0	
NON-BASIC $x_3 = x_4 = 0$		$z = 12$	0	0	1/2	3/4	0	$\leftarrow \Delta_j$

Thus optimum solution is obtained as : $x_1 = 3$, $x_2 = 1$, max $z = 12$.

EXAMINATION PROBLEMS

1. For the linear programming problem : Max. $z = 5x_1 + 3x_2$, subject to $3x_1 + 5x_2 \leq 15$, $5x_1 + 2x_2 \leq 10$, and $x_1, x_2 \geq 0$. find an optimal solution. Hence find how far the component c_1 of the vector **c** of the function $z = \mathbf{cx}$ can be increased without destroying the optimality of the solution.
2. In a linear programming problem, Max. $z = \mathbf{cx}$, $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, obtain the variations in c_j which are permitted without changing the optimal solution. Find this for the problem.

   Max. $z = 3x_1 + 5x_2$, subject to $x_1 + x_2 \leq 1$, $2x_1 + 3x_2 \leq 1$, and $x_1, x_2 \geq 0$

   [Meerut (B.Sc.) 90]

   [Ans. $x_1 = 0$, $x_2 = 1$, max. $z = 5$; $\Delta c_1 \leq 2$ and $-2 \leq \Delta c_2$.]

3. The following table gives the optimal solution to a linear programming problem.

Table 9.5

		$c_j \rightarrow$	2	3	1	0	0	
BASIC VAR.	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
x_1	2	1	1	0	1/2	4	− 1/2	
x_2	3	2	0	1	1	− 1	2	
		$z = 8$	0	0	3	5	5	$\leftarrow \Delta_j$

How much c_3 can be increased before the present basic solution will no longer be optimal. If c_3 is increased indefinitely, find the sequence of optimal basis.

4. Solve the following L.P.P. by (i) Graphical method and (ii) Simplex method :

Max. $z = 3x_1 + 5x_2$, subject to the constraints : $3x_1 + 2x_2 \leq 18$, $x_1 \leq 4$, $x_2 \leq 6$, and x_1 , $x_2 \geq 0$.

If the cost coefficient of x_1 is kept fixed, find the range for the cost coefficient of x_2 without affecting the optimal solution. **[IAS (Maths. 96]**

5. Given a linear programming problem : Maximize $z = 3x_2 + 5x_1$ subject to : $3x_1 + 5x_2 \leq 15$, $5x_1 + 2x_2 \leq 10$, x_1 , x_1 , $x_2 \geq 0$.
   (i) Find an optimal solution.
   (ii) Hence find how far the component c_1 of the vector **c** of the function $z = \mathbf{cx}$ can be increased without destroying the optimality of the solution.

   [Ans. (i) $x_1 = 20/19$, $x_2 = 45/19$, max. $z = 235/19$; (ii) $-16/5 \leq \Delta c_1 < 5/2$]

9.5. CHANGE IN THE COMPONENT 'b_l' OF VECTOR b

If any component (say, b_l) of vector $\mathbf{b} = (b_1 , b_2 , \ldots , b_l , \ldots , b_m)$ is changed, the optimality conditions $\Delta_j \equiv z_j - c_j \geq 0$ are not affected for the optimal solution $\mathbf{x_B}$ of $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \geq \mathbf{0}$.

On the other hand, any change in vector **b** will affect the optimal solution $\mathbf{x_B}$ $(\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0})$.

Furthermore, the feasibility of the new solution $\hat{\mathbf{x}}_\mathbf{B}$ can also be affected because of arbitrary changes in vector **b**.

Suppose that a component b_l of vector **b** is changed to $b_l + \Delta b_l$ so that the new vector $\hat{\mathbf{b}}$ becomes

$$\hat{\mathbf{b}} = [b_1 , b_2 , \ldots , (b_l + \Delta b_l) , \ldots , b_m] \qquad \ldots(9\cdot10)$$

Now, our aim is to find out those limits under which Δb_l can vary so that the feasibility of new solution $\hat{\mathbf{x}}_\mathbf{B} = \mathbf{B}^{-1}\hat{\mathbf{b}}$ is preserved and **B** remains the optimal basis.

Let
$$\mathbf{B}^{-1} = (\beta_1 , \beta_2 , \ldots , \beta_l , \ldots , \beta_m) = \begin{bmatrix} \beta_{11} & \beta_{12} \cdots \beta_{1l} \cdots \beta_{1m} \\ \beta_{21} & \beta_{22} \cdots \beta_{2l} \cdots \beta_{2m} \\ \vdots & \vdots \quad \vdots \quad \vdots \\ \beta_{i1} & \beta_{i2} \cdots \beta_{il} \cdots \beta_{im} \\ \vdots & \vdots \quad \vdots \quad \vdots \\ \beta_{m1} & \beta_{m2} \cdots \beta_{ml} \cdots \beta_{mm} \end{bmatrix}$$

Now, from (9·10) , the new right-hand side of $\mathbf{AX} = \mathbf{b}$ becomes

$$\hat{\mathbf{b}} = [b_1 , b_2 , \ldots , b_l , \ldots , b_m] + [0, 0, \ldots , \Delta b_l , \ldots , 0] = \mathbf{b} + [0, 0, \ldots, \Delta b_l , \ldots , 0] \qquad \ldots(9\cdot11)$$

Therefore,

$$\hat{\mathbf{x}}_\mathbf{B} = \mathbf{B}^{-1}\hat{\mathbf{b}} = \mathbf{B}^{-1}[\mathbf{b} + (0, 0, \ldots , \Delta b_l , \ldots , 0)] = \mathbf{B}^{-1}\mathbf{b} + \mathbf{B}^{-1}(0, 0, \ldots, \Delta b_l , \ldots , 0)$$

or
$$\hat{\mathbf{x}}_\mathbf{B} = \mathbf{x_B} + (\beta_1 , \beta_2 , \ldots , \beta_l , \ldots , \beta_m)(0, 0, \ldots , \Delta b_l , \ldots , 0) = \mathbf{x_B} + \beta_l \Delta b_l$$

or
$$\hat{x}_{Bi} = x_{Bi} + \beta_{il}\Delta b_l \qquad \ldots(9\cdot12)$$

Since the feasibility of $\hat{\mathbf{x}}_\mathbf{B}$ is preserved, from (9·12) we have

$$\left.\begin{array}{r} x_{Bi} + \beta_{il}\Delta b_l \geq 0 \\ \text{and} \quad x_{Bi} \geq 0 \end{array}\right\} \; i = 1, 2, \ldots , m \qquad \ldots(9\cdot13)$$

Letting $d_j = x_{Bi}$, $\lambda = \Delta b_l$ and $d_j' = \beta_{il}$, the result (9·1) of *lemma* gives

$$\boxed{\max_{\substack{i \\ \beta_{il} > 0}} \left[-\frac{x_{Bi}}{\beta_{il}} \right] \leq \Delta b_l \leq \min_{\substack{i \\ \beta_{il} < 0}} \left[\frac{-x_{Bi}}{\beta_{il}} \right]} \qquad \ldots(9{\cdot}14)$$

The following numerical examples will make the above discussion clear.

9.6. ILLUSTRATIVE EXAMPLES

Example 4. *The optimal Table 9·1 for the linear programming problem :*

$$\text{Max. } z = 3x_1 + 5x_2 + 4x_3\,, \qquad \text{subject to}$$

$$2x_1 + 3x_2 \leq 8,\quad 2x_2 + 5x_3 \leq 10,\quad 3x_1 + 2x_2 + 4x_3 \leq 15,\quad \text{and} \quad x_1\,, x_2\,, x_3 \geq 0\,.$$

is given in the example of Section 9·4 . Find the range over which b_2 can be changed maintaining the feasibility of the solution.

Solution. From *Table* 9·1 , we have

$$\mathbf{B} = (\mathbf{X}_2\,, \mathbf{X}_3\,, \mathbf{X}_1)\,, \mathbf{X_B} = \left(\frac{50}{41}, \frac{62}{41}, \frac{89}{41} \right),\ \mathbf{b} = (8, 10, 15),$$

and

$$\mathbf{B}^{-1} = \begin{bmatrix} 15/41 & 8/41 & 10/41 \\ -6/41 & 5/41 & 4/41 \\ -2/41 & -12/41 & 15/41 \end{bmatrix}$$

Letting $\Delta b_l = \Delta b_2$, gives

$$\max_{\substack{i \\ \beta_{i2} > 0}} \left[\frac{-50/41}{8/41}, \frac{-62/41}{5/41} \right] \leq \Delta b_2 \leq \min_{\substack{i \\ \beta_{i2} < 0}} \left[\frac{-89/41}{-12/41} \right]$$

or

$$\max. \left[-\frac{50}{8}, -\frac{62}{5} \right] \leq \Delta b_2 \leq \min. \left[\frac{89}{12} \right], \ i.e. \ -\frac{25}{4} \leq \Delta b_2 \leq \frac{89}{12}\,.$$

Example 5. *Given the following linear programming problem :*

$$\text{Max. } z = -x_1 + 2x_2 - x_3\,, \ \text{subject to}$$

$$3x_1 + x_2 - x_3 \leq 10,\quad -x_1 + 4x_2 + x_3 \geq 6,\quad x_2 + x_3 \leq 4,\quad \text{and} \quad x_1\,, x_2\,, x_3 \geq 0.$$

(a) *Determine an optimum solution to the problem.*

(b) *Determine the ranges for discrete changes in the components b_2 and b_3 of the required vector so as to maintain the optimality of the current optimum solution.*

(c) *Determine the effect of discrete changes in those components of the cost vector which correspond to basic variable.*

[Meerut M.Sc. (L.P.) 93]

Solution. (a) Introducing the slack variables $x_4 \geq 0$, $x_5 \geq 0$ in the first and third constraints, surplus variable $x_6 \geq 0$ together with an artificial variable $a_1 \geq 0$ in the second constraint, and then solving the resulting problem by usual simplex method, the following optimum simplex table is obtained :

Table 9·6 . Optimum Table

		$c_j \rightarrow$	-1	2	-1	0	0	0	$-M$	
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$ (β_1)	$\mathbf{X}_5$ (β_2)	$\mathbf{X}_6$	$\mathbf{A}_1$ (β_3)	
x_4	0	6	3	0	-2	1	-1	0	0	
x_2	2	4	0	1	1	0	1	0	0	
x_6	0	10	1	0	3	0	4	1	-1	
	$z = 8$		1	0	3	0	2	0	M	$\leftarrow \Delta_j$

Thus, optimum solution is : $x_1 = 0$, $x_2 = 4$, $x_3 = 0$, max. $z = 8$. **Ans.**

(b) The individual effects of changes in b_2 and b_3 , where $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{B}^{-1} = (\beta_1, \beta_2, \beta_3) = (\mathbf{X_4}, \mathbf{X_5}\ \mathbf{A_1})$ such that the optimality of the basic feasible solution is not violated, are given by

$$\max_{\substack{i \\ \beta_{il} > 0}} \left[\frac{-x_{Bi}}{\beta_{il}}\right] \le \Delta b_l \le \min_{\substack{i \\ \beta_{il} < 0}} \left[\frac{-x_{Bi}}{\beta_{il}}\right]$$

For, $l = 2$, consider the second column of $\mathbf{B}^{-1}$ to get

$$-5/2 \le \Delta b_2 \le \left(\frac{-6}{-1}\right); \text{ or } -5/2 \le \Delta b_2 \le 6$$

and for $l = 3$, consider *third column* (*i.e.* $\mathbf{X}_6$) of $\mathbf{B}^{-1}$ to get

$$\text{Max.} \left[\frac{-10}{-1}\right] \le \Delta b_3 \text{ or } \Delta b_3 \ge 10.$$

Thus $-5/2 \le \Delta b_2 \le 6$ and $\Delta b_3 \ge 10$. **Ans.**

(c) For the change in the component of **c** corresponding to basic variables, use the relationship :

$$\max_{\substack{j \\ x_{rj} > 0}} \left[\frac{-\Delta_j}{x_{rj}}\right] \le \Delta c_{Br} \le \min_{\substack{j \\ x_{rj} < 0}} \left[\frac{-\Delta_j}{x_{rj}}\right]$$

Therefore, we have

(*i*) $\max\left[\frac{-3}{1}, \frac{-2}{1}\right] \le \Delta c_2$ or $\Delta c_2 \ge -2$

(*ii*) $\max\left[\frac{-1}{3}\right] \le \Delta c_4 \le \min\left[\frac{-3}{-2}, \frac{-2}{-1}\right]$ or $-1/3 \le \Delta c_4 \le 3/2$

(*iii*) $\max\left[\frac{-1}{1}, \frac{-3}{3}, \frac{-2}{4}\right] \le \Delta c_5$ or $\Delta c_5 \ge -1/2$.

Example 6. (a) *Discuss the effect of discrete changes in the requirements (on the right side of the inequalities) for the following linear programming problem :*

Max. $z = 3x_1 + 4x_2 + x_3 + 7x_4$, *subject to the constraints* :

$8x_1 + 3x_2 + 4x_3 + x_4 \le 7$, $2x_1 + 6x_2 + x_3 + 5x_4 \le 3$, $x_1 + 4x_2 + 5x_3 + 2x_4 \le 8$, *and* $x_1, x_2, x_3, x_4 \ge 0$.

[Meerut M.Sc. (Math.) 92]

(b) *Discuss the effect of discrete changes in* **c** *on the optimality of an optimum basic feasible solution to the above LP.P..*

Solution. Introducing slack variables x_5, x_6 and x_7, we get the following **starting table :**

Starting Table

		$c_j \rightarrow$	3	4	1	7	0	0	0
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{X_6}$	$\mathbf{X_7}$
x_5	0	7	8	3	4	1	1	0	0
x_6	0	3	2	6	1	5	0	1	0
x_7	0	8	1	4	5	2	0	0	1
	$z = 0$		-3	-4	-1	-7	0	0	0

Performing usual simplex routine we get the following ***optimal table :***

Optimal Table

		$c_j \rightarrow$	3	4	1	7	0	0	0
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{X_6}$	$\mathbf{X_7}$
x_1	3	16/19	1	9/38	1/2	0	5/38	$-1/38$	0
x_4	7	5/19	0	21/19	0	1	$-1/19$	8/38	0
x_7	0	126/19	0	59/38	9/2	0	$-1/38$	$-15/38$	1
	$z = 83/19$		0	169/38	1/2	0	1/38	53/38	0

(a) From above optimal table, we have

$$\mathbf{X_B} = \left[\frac{16}{19}, \frac{5}{19}, \frac{126}{19}\right], \; \mathbf{b} = [7, 3, 8], \quad \text{and} \quad \mathbf{B}^{-1} = (\mathbf{X}_5, \mathbf{X}_6, \mathbf{X}_7) = \begin{pmatrix} 5/38 & -1/38 & 0 \\ -1/19 & 8/38 & 0 \\ -1/38 & -15/38 & 1 \end{pmatrix}$$

Thus, the separate range of changes in b_1, b_2 and b_3 where $\mathbf{b} = (b_1, b_2, b_3)$ such that the new basic feasible solution still remains optimal, are given by

(i) $\frac{-16/19}{5/38} \le \Delta b_1 \le \min\left\{\frac{-5/19}{-1/19}, \frac{-126/19}{-1/38}\right\}$ or $-32/5 \le \Delta b_1 \le 5$;

(ii) $\frac{-5/19}{8/38} \le \Delta b_2 \le \min\left\{\frac{-16/19}{-1/38}, \frac{-126/19}{-15/38}\right\}$ or $-5/4 \le \Delta b_2 \le 84/5$;

(iii) $\frac{-126/19}{1} \le \Delta b_3$ or $-126/19 \le \Delta b_3$, having no upper bound for discrete change Δb_3 in the component b_3.

Thus, above limits for changes in $\mathbf{b}$ will specify the required ranges so that every new basic solution will remain optimum feasible.

(b) The cost vector corresponding to the basis is $\mathbf{C_B} = (3, 7, 0)$.

If we write the cost vector as $\mathbf{C} = (c_1, c_2, c_3, c_4, c_5, c_6, c_7)$ and the basis cost vector as $\mathbf{C_B} = (c_1, c_4, c_7)$, then one can classify the changes in c_j's in two ways:

(i)change in c_j when $j = 1, 4, 7$; and (ii) change in c_j when $j = 2, 3, 5, 6$.

For (i), we know that an optimum basic feasible solution will maintain its optimality if the change Δc_k in c_k satisfies:

$$\boxed{\underset{x_{kj} > 0}{\textbf{Max}} \left\{\frac{-\Delta_j}{x_{kj}}\right\} \le \Delta c_k \le \underset{x_{kj} < 0}{\textbf{Min}} \left\{-\frac{\Delta_j}{x_{kj}}\right\}}$$

Thus the required limits for changes in c_1, c_4, c_7 are given by

(i) $\text{Max.}\left[\frac{-169/38}{9/38}, \frac{-1/2}{1/2}, \frac{-1/38}{5/38}\right] \le \Delta c_1 \le \frac{-53/38}{-1/38}$ or $-1/5 \le \Delta c_1 \le 53$;

(ii) $\text{Max.}\left[\frac{-169/38}{21/19}, \frac{-53/38}{8/38}\right] \le \Delta c_4 \le \frac{-1/38}{-1/19}$ or $\frac{-169}{42} \le \Delta c_4 \le 1/2$;

and *(iii)* $\text{Max.}\left[\frac{-169/38}{59/38}, \frac{-1/2}{9/2}\right] \le \Delta c_7 \le \text{Min}\left[\frac{-1/38}{-1/38}, \frac{-53/38}{-15/38}\right]$ or $-1/9 \le \Delta c_7 \le 1$.

For (ii), we know that the change Δc_k in c_k must satisfy the upper limit $\Delta c_k \le (z_k - c_k)$ in order to maintain the optimality of the optimum basic feasible solution. Thus, we have

$$\Delta c_2 \le 169/38, \quad \Delta c_3 \le 1/2, \; \Delta c_5 \le 1/38, \text{ and } \Delta c_6 \le 53/38.$$

Here we observe that there are no lower limits for changes in the components c_2, c_3, c_5 and c_6.

EXAMINATION PROBLEMS

1. Consider the LP problem : Max. $z = -x_2 + 3x_3 - 2x_5$ subject to

$$3x_2 - x_3 + 2x_5 \le 7, -2x_2 + 4x_3 \le 12, -4x_2 + 3x_3 + 8x_5 \le 10, \text{ and } x_2, x_3, x_5 \ge 0.$$

The optimal table of this problem is given below :

		$c_j \to$	0	−1	3	0	−2	0	
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{X}_6$	
x_2	−1	4	2/5	1	0	1/10	4/5	0	
x_3	−3	5	1/5	0	1	3/10	2/5	0	
x_6	0	11	1	0	0	−1/2	10	1	
	$z = 11$		1/5	0	0	4/5	12/5	0	$\leftarrow \Delta_j$

(a) Formulate the dual problem for this primal problem.
(b) What are the optimal values of dual variables ?

(c) How much c_5 be decreased before x_2 goes into basis ?

(d) How much can the '7' in first constraint be increased before the basis would change ?

[Ans. (a) Min $z_w = 7w_1 + 12w_2 + 10w_3$, subject to the constraints :

$-3w_1 + 2w_2 + 4w_3 \le 1$, $-w_1 + 4w_2 + 3w_3 \ge 3$, $-2w_1 - 8w_3 \le 2$, w_1 , w_2 and w_3 are non-negative.

(b) $w_1 = 1/5$, $w_2 = /4/5$, $w_3 = 0$.

(c) change c_5 to $c_5 + \Delta c_5$ such that $\Delta c_5 \ge 12/5$

(d) 7 changes to $7 + \Delta b_1$ such that $\Delta b_1 \le -10$.]

2. Consider the LP problem : Max. $z = 2x_1 + x_2 + 4x_3 - x_4$ subject to the constraints :

$x_1 + 2x_2 + x_3 - 3x_4 \le 8$, $-x_2 + x_3 + 2x_4 \le 0$, $2x_1 + 7x_2 - 5x_3 - 10x_4 \le 21$, and x_1 , x_2 , x_3 , $x_4 \ge 0$.

The optimum solution to this problem is contained in the following simplex table.

		$c_j \rightarrow$	2	1	4	−1	0	0	0
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{X_6}$	$\mathbf{X_7}$
x_1	2	8	1	0	3	1	1	2	0
x_2	1	0	0	1	−1	−2	0	−1	0
x_7	0	5	0	0	−4	2	−2	3	1
	$z = 16$		0	0	1	1	2	3	0

For each of the discrete parameter changes listed below, make the necessary corrections in the optimum table and solve the resulting problem :

(a) change c_1 to 1, (b) change **b** to (3, − 2, 4) (c) change b_2 to 11 (d) change **C** to (1, 2, 3, 4)

[Hint. (a) c_1 changes from 2 to 1, and $c_1 \in \mathbf{C_B}$. Revise the optimal table accordingly and compute new Δ_j**'s** as (0, 0, −2, 0, 1, 1, 0). Obtain the new optimum solution by improving the simplex table to get the new optimal solution ; $x_1 = 0$, $x_2 = 8/3$, $x_3 = 8/3$, $x_4 = 0$ and max $z = 40/3$]

(b) When **b** changes from (8 0 21) to (3 −2 4) the new values of current basic variables are obtained by $\mathbf{X_B} = \mathbf{B}^{-1}\,\mathbf{b}$ as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -8 \end{pmatrix}$$

Since current basic solution is optimal but infeasible, therefore use dual simplex method.

It will be observed that all elements of the departing row are non-negative, by dual simplex method, the given problem does not possess any feasible solution.

(c) By dual simplex method new optimal solution is obtained as : $x_1 = 49/2$, $x_2 = x_3 = 0$, $x_4 = 11/2$, max $z = 87/2$.

(d) Using simplex method obtain the new optimum solution : $x_1 = 0$, $x_2 = 21/2$, $x_3 = 11/10$, $x_4 = 47/10$ and max. $z \equiv 431/10$].

3. Find out the limits of variations of the costs c_1 , c_2 , c_3 , c_4 , c_5 , c_6 and the second element b_2 of requirement vector of the following L.P.P. for which the optimal solution remains optimal :

Max. $z = -x_2 + 3x_3 - 2x_5$ subject to

$$\begin{aligned} x_1 + 3x_2 - x_3 \quad + 2x_5 \quad &= 7 \\ -2x_2 + 4x_3 + x_4 \quad &= 12 \\ -4x_2 + 3x_3 \quad + 8x_5 + x_6 &= 10 \end{aligned}$$

and $x_j \ge 0$, $j = 1, 2, ..., 6$. **[Meerut M.Sc. (Maths.) 94]**

[Hint : First find the optimal solution, Then proceed as solved examples given earlier.]

4. Discuss the effect of discrete changes in the requirement vector on the optimal solution of the following linear programming problem :

Max. $z = 3x_1 - x_2 - 2x_3$, subject to $-x_1 + 3x_2 + 2x_3 \le 7$, $4x_1 - 2x_2 \le 12$, $3x_1 - 4x_2 + 8x_3 \le 18$; x_1 , x_2 , $x_3 \ge 0$.

[Delhi B.Sc. (Maths.) 93]

[Ans. $-3 \le \Delta b_1 \le 45$, $-18 \le \Delta b_2 \le 6$]

5. For the following L.P.P. find the separate ranges of variations of right hand sides of the constraints consistent with the optimal solution :

Max. $z = -x_1 + 2x_2 - x_3$, subject to $3x_1 + x_2 - x_3 \le 10$, $-x_1 + 4x_2 + x_3 \le 6$, $x_2 + x_3 \le 4$, x_1 , x_2 , $x_3 \ge 0$.

[Hint. Proceed as solved example]

6. Determine the optimal solution to the following linear programming problem :

Max. $z = 15x_1 + 10x_2$, subject to $5x_1 + 3x_2 \le 200$, $2x_1 + 2x_2 \le 100$, x_1 , $x_2 \ge 0$.

Find the range for the discrete changes in the first component of the requirement vector so as to maintain the optimality of the current optimal solution.

7. Following is the optimal table of an LPP. :

BASIC VAR.	X_B	X_1	X_2	S_1	A_1	A_2	S_2	
x_1	3/5	1	0	1/5	3/5	−1/5	0	
x_2	6/5	0	1	−3/5	−4/5	3/5	0	
s_2	0	0	0	1	1	−1	1	
	$z = 12/5$	0	0	1/5	$M-\frac{2}{5}$	$M-\frac{1}{5}$	0	$\leftarrow \Delta_j$

Using sensitivity analysis find the optimal solution to the new problem obtained by (i) changing **c** to **c** + **c′**, where **c′** = (− 5, 10)^T; (ii) changing **b** to **b** + **b′**, where **b′** = (3, 4, 1)^T. **[Roorkee M.Sc.I (OR) 96]**

9.7. CHANGE IN THE COMPONENT 'a_{ij}' OF MATRIX A

Let us assume that the component a_{ij} in the ith row and jth column of matrix **A** is changed to $(a_{ij} + \Delta a_{ij})$. Furthermore, such variation can be made in two possible ways :

(i) only that a_{ij} is changed to $a_{ij} + \Delta a_{ij}$ which does not belong to the basis matrix **B** ,

(ii) only that a_{ij} is changed to $a_{ij} + \Delta a_{ij}$ which belongs to the basis matrix **B**.

We now discuss these two cases one by one.

Case 1. (Variation in a_{ij} when $a_{ij} \notin$ B)

If $a_{ij} \notin \mathbf{B}$, then the change in such a_{ij} also does not affect the optimal solution $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$.

Thus, it only remains to observe the effect on the optimal conditions (*viz.* $\hat{z}_j - c_j \geq 0$) when $a_{ij} \notin \mathbf{B}$ is changed to $(a_{ij} + \Delta a_{ij})$,

Let $\mathbf{B}^{-1} = [\beta_1, \beta_2, \dots, \beta_m]$ and $\mathbf{a_j} = [a_{1j}, a_{2j}, \dots, a_{mj}]$

Therefore,

$$\hat{\mathbf{a}}_j = [a_{1j}, a_{2j}, \dots, (a_{ij} + \Delta a_{ij}), \dots, a_{mj}] = [a_{1j}, a_{2j}, \dots, a_{ij}, \dots, a_{mj}] + [0, 0, \dots, \Delta a_{ij}, \dots, 0]$$
$$= \mathbf{a_j} + [0, 0, \dots, \Delta a_{ij}, \dots, 0]$$

and

$$\hat{z}_j = \mathbf{C_B B}^{-1}\hat{\mathbf{a}}_j = \mathbf{C_B B}^{-1}[\mathbf{a_j} + (0, 0, \dots, \Delta a_{ij}, \dots, 0)]$$
$$= \mathbf{C_B B}^{-1}\mathbf{a_j} + \mathbf{C_B B}^{-1}[0, 0, \dots, \Delta a_{ij}, \dots, 0]$$
$$= z_j + \mathbf{C_B}(\beta_1, \beta_2, \dots, \beta_i, \dots, \beta_m)(0, 0, \dots, \Delta a_{ij}, \dots, 0)$$
$$= z_j + \mathbf{C_B}\beta_i \Delta a_{ij}.$$

Thus, the optimality conditions will remain satisfied if—

$$\hat{z}_j - c_j \geq 0 \quad \text{or} \quad [z_j + \mathbf{C_B}\beta_i \Delta a_{ij}] - c_j \geq 0$$

$$\left[\begin{array}{ll} & (z_j - c_j) + \Delta a_{ij}(\mathbf{C_B}\beta_i) \geq 0 \\ \text{and, originally} & z_j - c_j \geq 0 \end{array}\right] \qquad \text{...(9·15)}$$

Since the conditions for using the result (9·1) are satisfied in (9·5), so we have

$$\boxed{\underset{\mathbf{C_B}\beta_i > 0}{\underset{i}{\text{Max.}}}\left[\frac{-\Delta_j}{\mathbf{C_B}\beta_i}\right] \leq \Delta a_{ij} \leq \underset{\mathbf{C_B}\beta_i < 0}{\underset{i}{\text{Min}}}\left[\frac{-\Delta_j}{\mathbf{C_B}\beta_i}\right].} \qquad \text{...(9·16)}$$

Also, Δa_{ij} is unrestricted if $\mathbf{C_B}\beta_i = 0$.

In this case, neither the optimal solution $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$ nor the value of the objective function $(z = \mathbf{C_B X_B})$ changes.

The following example illustrates the above discussion.

Example 7. *Now, from the example (Table 9·1) which is continuing in this chapter, we have*

$\mathbf{a_4} \notin \mathbf{B}$, $\mathbf{B} = (\mathbf{X_2}, \mathbf{X_3}, \mathbf{X_1})$,

and

$$\mathbf{B}^{-1} = (\mathbf{X_4}, \mathbf{X_5}, \mathbf{X_6}) = (\beta_1, \beta_2, \beta_3) = \begin{bmatrix} 15/41 & 8/41 & -10/41 \\ -6/41 & 5/41 & 4/41 \\ -2/41 & -12/41 & 15/41 \end{bmatrix}, \mathbf{C_B} = [5, 4, 3].$$

We compute

$$C_B\beta_1 = (5\ 4\ 3)\left(\frac{15}{41}, \frac{-6}{41}, \frac{-2}{41}\right) = \frac{45}{41},\quad C_B\beta_2 = (5\ 4\ 3)\left[\frac{8}{41}, \frac{5}{41}, \frac{-12}{41}\right] = \frac{24}{41}$$

$$C_B\beta_3 = (5\ 4\ 3)\left[\frac{-10}{41}, \frac{4}{41}, \frac{15}{41}\right] = \frac{11}{41}.$$

Now, using the result (9·16) for $j = 4$, we have

$$\frac{-\Delta_4}{C_B\beta_1} \le \Delta a_{14} < \infty \text{ (since all } C_B\beta_i > 0) \qquad \text{or} \qquad -\frac{45/41}{45/41} \le \Delta a_{14} < \infty$$

or $$-1 \le \Delta a_{14} < \infty. \qquad ..(i)$$

Similarly, $$\frac{-\Delta_3}{C_B\beta_2} \le \Delta a_{24} < \infty \qquad \text{or} \qquad -1 \le \Delta a_{24} < \infty \qquad ...(ii)$$

and $$-1 \le \Delta a_{34} < \infty. \qquad ...(iii)$$

Case 2. (Variation in $a_{ij} \in$ B)

If $a_{ij} \in \mathbf{B}$ is changed to $a_{ij} + \Delta a_{ij}$, then the basis matrix $\mathbf{B}$ will certainly change. Consequently, $\mathbf{X_B} = \mathbf{B}^{-1}\mathbf{b}$ and $z_j = \mathbf{C_B B}^{-1}\mathbf{a_j}$ will also change.

Let $\hat{\mathbf{B}}$ be the new basis matrix obtained after $a_{ij} \in \mathbf{B}$ is changed to $a_{ij} + \Delta a_{ij}$.

Then, we shall first find out $\hat{\mathbf{B}}^{-1}$.

Let $\mathbf{B} = (\mathbf{b_1}, \mathbf{b_2}, \dots, \mathbf{b_p}, \dots, \mathbf{b_m})$ and $\mathbf{B}^{-1} = (\beta_1, \beta_2, \dots, \beta_m)$.

If $\mathbf{a_j} = \mathbf{b_p}$, then, $(a_{1j}, a_{2j}, \dots, a_{ij}, \dots, a_{mj}) = (b_{1p}, b_{2p}, \dots, b_{ip}, \dots, b_{mp})$ (since $a_{ij} = b_{ip}$)

and $$a_{ij} + \Delta a_{ij} = b_{ip} + \Delta a_{ij}. \qquad ...(9·17)$$

Now, the new basis matrix $\hat{\mathbf{B}}$ is given by $\hat{\mathbf{B}} = [\mathbf{b_1}, \mathbf{b_2}, \dots, \hat{\mathbf{b}}_\mathbf{p}, \dots, \mathbf{b_m}]$

$$\hat{\mathbf{b}}_\mathbf{p} = [b_{1p}, b_{2p}, \dots, b_{ip} + \Delta a_{ij}, \dots, b_{mp}]. \qquad ...(9·18)$$

Since $\hat{\mathbf{b}}_\mathbf{p} \notin \mathbf{B}$ can be expressed as the linear combination of vectors $\mathbf{b_1}, \mathbf{b_2}, \dots, \mathbf{b_p}, \dots, \mathbf{b_m}$, in $\mathbf{B}$, therefore

$$\hat{\mathbf{b}}_\mathbf{p} = \lambda_1\mathbf{b_1} + \lambda_2\mathbf{b_2} + \dots + \lambda_p\mathbf{b_p} + \dots + \lambda_m\mathbf{b_m} = (\lambda_1, \lambda_2, \dots, \lambda_p, \dots, \lambda_m)(\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, \dots, \mathbf{b_m})$$
$$= \mathbf{B}\bar{\lambda} \text{ where } \bar{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_m). \qquad ...(9·19)$$

Thus, from (9·19)
$$\bar{\lambda} = \mathbf{B}^{-1}\hat{\mathbf{b}}_\mathbf{p}$$
$$= (\beta_1, \beta_2, \dots, \beta_i, \dots, \beta_m)(b_{1p}, b_{2p}, \dots, b_{ip} + \Delta a_{ij}, \dots, b_{mp}) \qquad \text{[from (9·18)]}$$
$$= \sum_{i=1}^{m} \beta_i b_{ip} + \beta_i \Delta a_{ij} = \mathbf{B}^{-1}\mathbf{b_p} + \beta_i \Delta a_{ij}$$

or $$\bar{\lambda} = \mathbf{e_p} + \beta_i \Delta a_{ij} \qquad ...(9·20)$$

$p \qquad m-p$

[Since $\mathbf{Be_p} = [\mathbf{b_1}, \mathbf{b_2}, \dots, \mathbf{b_p}, \dots, \mathbf{b_m}](0, 0, \dots, 1, 0, 0, \dots, 0) = \mathbf{b}_p$, therefore $\mathbf{B}^{-1}\mathbf{b}_p = \mathbf{e}_p$]

Now, from (9·20), we have $\qquad p \qquad m-p$

$$(\lambda_1, \lambda_2, \dots, \lambda_p, \dots, \lambda_m) = (0, 0, \dots, 1, 0, \dots, 0) + (\beta_{1i}, \beta_{2i}, \dots, \beta_{pi}, \dots, \beta_{mi})\Delta a_{ij}$$

Equating the pth element and kth element on both sides, we have

and $$\left\{\begin{array}{l} \lambda_p = 1 + \beta_{pi}\Delta a_{ij} \\ \lambda_k = 0 + \beta_{ki}\Delta a_{ij} \end{array}\right\} \qquad ...(9·21)$$

Now, $\hat{\mathbf{B}}^{-1}$ exist only when $\hat{\mathbf{B}}$ is non-singular (*i.e.* $|\hat{\mathbf{B}}| \ne 0$). So we must have $\lambda_p \ne 0$.

Therefore, from (9·21)

$$1 + \beta_{pi}\Delta a_{ij} \ne 0, \text{ or } \Delta a_{ij} \ne -1/\beta_{pi}. \qquad ...(9·22)$$

Now, we can introduce a new matrix $\mathbf{E}$ called the elementary matrix, which differs from an identity matrix $\mathbf{I_m}$ in the pth column only.

The pth column of matrix $\mathbf{E}$ can be obtained from (9·19) as follows :

$$\hat{\mathbf{b}}_p = (\lambda_1 \mathbf{b}_1 + \lambda_2 \mathbf{b}_2 + \dots + \lambda_p \mathbf{b}_p + \dots + \lambda_m \mathbf{b}_m)$$

or
$$\mathbf{b}_p = -\frac{\lambda_1}{\lambda_p}\mathbf{b}_1 - \frac{\lambda_2}{\lambda_p}\mathbf{b}_2 - \dots + \frac{\hat{\mathbf{b}}_p}{\lambda_p} - \dots - \frac{\lambda_m}{\lambda_p}\mathbf{b}_m$$
$$= \left(-\frac{\lambda_1}{\lambda_p}, -\frac{\lambda_2}{\lambda_p}, \dots, +\frac{1}{\lambda_p}, - \dots, -\frac{\lambda_m}{\lambda_p}\right)(\mathbf{b}_1, \mathbf{b}_2, \dots, \hat{\mathbf{b}}_p, \dots \mathbf{b}_m)$$
$$= \bar{\eta}\,\hat{\mathbf{B}} \qquad \text{...(9·23)}$$

where $\bar{\eta} = \left(\frac{-\lambda_1}{\lambda_p}, -\frac{\lambda_2}{\lambda_p}, \dots, +\frac{1}{\lambda_p}, \dots, -\frac{\lambda_m}{\lambda_p}\right) =$ the pth column of $\mathbf{E}$.

Now,
$$\mathbf{E} = \begin{bmatrix} 1 & 0 & \dots & 0 & -\lambda_1/\lambda_p & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & -\lambda_2/\lambda_p & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -\lambda_3/\lambda_p & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 & -\lambda_{p-1}/\lambda_p & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1/\lambda_p & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -\lambda_{p+1}/\lambda_p & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & -\lambda_m/\lambda_p & 0 & \dots & 1 \end{bmatrix}$$
$$= (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{p-1}, \bar{\eta}, \mathbf{e}_{p+1}, \dots, \mathbf{e}_m).$$

Therefore, $\hat{\mathbf{B}}\mathbf{E} = \hat{\mathbf{B}}(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{p-1}, \bar{\eta}, \mathbf{e}_{p-1}, \dots, e_m) = (\hat{\mathbf{B}}\mathbf{e}_1, \hat{\mathbf{B}}\mathbf{e}_2, \dots, \hat{\mathbf{B}}\mathbf{e}_{p-1}, \hat{\mathbf{B}}\bar{\eta}, \hat{\mathbf{B}}\mathbf{e}_{p+1}, \dots, \hat{\mathbf{B}}\mathbf{e}_m)$
$= (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_{p-1}, \mathbf{b}_p, \mathbf{b}_{p+1}, \dots, \mathbf{b}_m)$

Since $\hat{\mathbf{B}}\bar{\eta} = \mathbf{b}_p$ from (9·23), and $\hat{\mathbf{B}}\mathbf{e}_1 = (\mathbf{b}_1, \mathbf{b}_2, \dots, \hat{\mathbf{b}}_p, \dots, \mathbf{b}_m)(1, 0, \dots, 0) = \mathbf{b}_1$, etc.

$\therefore$
$$\hat{\mathbf{B}}\,\mathbf{E} = \mathbf{B}. \qquad \text{...(9·24)}$$

From (9·24), we get
$$\hat{\mathbf{B}}(\mathbf{E}\mathbf{B}^{-1}) = \mathbf{B}\mathbf{B}^{-1} \Rightarrow (\hat{\mathbf{B}}^{-1}\hat{\mathbf{B}})\,\mathbf{E}\mathbf{B}^{-1} = (\hat{\mathbf{B}})^{-1}\mathbf{I} \Rightarrow (\hat{\mathbf{B}})^{-1} = \mathbf{E}\mathbf{B}^{-1}. \qquad \text{...(9·25)}$$

Thus, new solution becomes $\quad \hat{\mathbf{X}}_B = (\hat{\mathbf{B}})^{-1}\mathbf{b} = (\mathbf{E}\mathbf{B}^{-1})\mathbf{b} = \mathbf{E}\mathbf{X}_B$

and by matrix transformation formula, we have
$$\begin{cases} \hat{x}_{Bk} = x_{Bk} - \dfrac{\lambda_k}{\lambda_p} x_{Bp}, \ (k \neq p, k = 1, 2, \dots, m) \\ \hat{x}_{Bp} = \dfrac{x_{Bp}}{\lambda_p}, \ k = p. \end{cases}$$

Using (9·21) we get
$$\begin{cases} \hat{x}_{Bk} = x_{Bk} - \dfrac{\Delta a_{ij}\,\beta_{ki}}{1 + \Delta a_{ij}\,\beta_{pi}} x_{Bp}, \ (k \neq p, k = 1, 2, \dots, m) \\ \hat{x}_{Bp} = \dfrac{x_{Bp}}{1 + \Delta a_{ij}\,\beta_{pi}} \end{cases} \qquad \text{...(9·26)}$$

But, feasibility and optimality both should be preserved, Therefore, we find the limits of Δa_{ij} for feasibility and optimality separately. Then the intersection of the limits for both the conditions will be the required result.

(i) When $\hat{\mathbf{X}}_B$ is feasible : If $\hat{\mathbf{X}}_B$ preserves the feasibility, then we require $\hat{x}_{Bk} \geq 0$, $\hat{x}_{Bp} \geq 0$.

Therefore, from (9·26), we have
$$\left.\begin{aligned} x_{Bk} - \frac{\Delta a_{ij}\,\beta_{ki}}{1 + \Delta a_{ij}\,\beta_{pi}} x_{Bp} &\geq 0, \ k = 1, 2, 3, \dots, m, \quad k \neq p \\ \frac{x_{Bp}}{1 + \Delta a_{ij}\,\beta_{pi}} &\geq 0, \ k = p. \end{aligned}\right\} \qquad \text{...(9·27)}$$

If $x_{Bp} \neq 0$, then we must have
$$1 + \Delta a_{ij}\,\beta_{pi} > 0. \qquad \text{...(9·28)}$$

Also, we require from (9·27), that $x_{Bk}(1+\Delta a_{ij}\beta_{pi})-\Delta a_{ij}\beta_{ki}x_{Bp} \ge 0,\ k\neq p$

or
$$\left.\begin{aligned} x_{Bk}+\Delta a_{ij}[x_{Bk}\ \beta_{pi}-x_{Bp}\ \beta_{ki}] &\ge 0,\ k\neq p\\ x_{Bk} &\ge 0 \text{ (originally)}\end{aligned}\right\} \quad ...(9\cdot29)$$

The result (9·1) of *lemma* can be used by taking

$$d_j = x_{Bk},\ d_j' = x_{Bk}\,\beta_{pi}-x_{Bp}\,\beta_{ki} = \mathbf{P}\text{ (say)},\ \lambda=\Delta a_{ij}.$$

Thus, we shall have the range of Δa_{ij} as

$$\boxed{\underset{\substack{k\neq p\\ \mathbf{P}>0}}{\mathbf{Max}}\left[\frac{-x_{Bk}}{\mathbf{P}}\right] \le \Delta a_{ij} \le \underset{\substack{k\neq p\\ \mathbf{P}<0}}{\mathbf{Min}}\left[\frac{-x_{Bk}}{\mathbf{P}}\right] \textbf{ where } \mathbf{P}=x_{Bk}\,\beta_{pi}-x_{Bp}\,\beta_{ki}.} \quad ...(9\cdot30)$$

(ii) When optimality for $\hat{\mathbf{x}}_\mathbf{B}$ is preserved : For optimality, we must have $\hat{z}_q - c_q \ge 0$.

But,
$$\hat{z}_q - c_q = \mathbf{C_B}\hat{\mathbf{B}}^{-1}\mathbf{a_q}-c_q=\mathbf{C_B}(\mathbf{EB}^{-1})\,\mathbf{a_q}-c_q$$
$$= \mathbf{C_B}(\mathbf{E\,B}^{-1}\mathbf{a_q})-c_q=\mathbf{C_B}(\mathbf{EX_q})-c_q. \qquad \text{[from (9·25)]}$$

$$=(c_{B1},c_{B2},\ldots,c_{Bp},\ldots,c_{Bm})\begin{bmatrix} x_{1q}-\dfrac{\lambda_1}{\lambda_p}x_{pq}\\ \vdots\\ x_{p-1,q}-\dfrac{\lambda_{p-1}}{\lambda_p}x_{pq}\\ \dfrac{x_{pq}}{\lambda_p}\\ x_{p+1,q}-\dfrac{\lambda_{p+1}}{\lambda_p}x_{pq}\\ \vdots\\ x_{mq}-\dfrac{\lambda_m}{\lambda_p}x_{pq}\end{bmatrix}-c_q$$

$$=\sum_{k=1,k\neq p}^{m} c_{Bk}x_{kq}-\frac{x_{pq}}{\lambda_p}\sum_{k=1,k\neq p}^{m}c_{Bk}\lambda_k+\frac{x_{pq}\,c_{Bp}}{\lambda_p}-c_q$$

or
$$\hat{z}_q-c_q=(z_q-c_q)-\frac{\Delta a_{ij}\,x_{pq}}{1+\Delta a_{ij}\,\beta_{pi}}\,\mathbf{C_B}\beta_i \qquad \text{(after simplification).}$$

We now require for optimality that $(z_q-c_q)-\dfrac{\Delta a_{ij}\,x_{pq}}{1+\Delta a_{ij}\,\beta_{pi}}\,\mathbf{C_B}\,\beta_i \ge 0$, *i.e.*

or
$$(z_q-c_q)(1+\Delta a_{ij}\,\beta_{pi})-\Delta a_{ij}x_{pq}\,\mathbf{C_B}\beta_i \ge 0$$

and
$$\left.\begin{aligned}(z_q-c_q)+\Delta a_{ij}[\beta_{pi}(z_q-c_q)-x_{pq}\,\mathbf{C_B}\beta_i] &\ge 0\\ z_q-c_q &\ge 0 \text{ (originally)}\end{aligned}\right\} \quad (9.31)$$

Using the result (9·1) of ***Lemma*** by taking $d_j=z_q-c_q$, $\lambda=\Delta a_{ij}$, $d_j'=\beta_{pi}(z_q-c_q)-x_{pq}\,\mathbf{C_B}\,\beta_i=\mathbf{P}$ (say), we get

$$\boxed{\underset{\substack{q\\ \mathbf{P}>0}}{\mathbf{Max.}}\left[\frac{-\Delta_q}{\mathbf{P}}\right]\le \Delta a_{ij}\le \underset{\substack{q\\ \mathbf{P}<0}}{\mathbf{Min.}}\left[\frac{-\Delta_q}{\mathbf{P}}\right] \textbf{ where } \Delta_q=z_q-c_q.} \quad ...(9\cdot31)$$

Finally, the required limits for Δa_{ij} are contained in the intersection of (9·30) and (9·32).

Q. 1. Find the limits of variation of a_{ik} so that the optimal feasible solution of $\boldsymbol{Ax=b}$, $\boldsymbol{x}\ge\mathbf{0}$, max $z=\mathbf{cx}$ remains optimal feasible solution, when $\mathbf{a_k}$ is not a vector of the optimal base.

2. Discuss the changes in the components a_{ij} of the vector $\mathbf{a_j}\in\mathbf{B}$ for the given L.P. problem : max $z=\mathbf{cx}$, subject to $\boldsymbol{Ax}=\mathbf{b}$, $\mathbf{x}\ge\mathbf{0}$.

Example 8. *The optimal solution of the linear programming problem :*

Max. $z=10x_1+3x_2+6x_3+5x_4$, *subject to the constraints*

$$x_1 + 2x_2 + x_4 \le 6, \quad 3x_1 + 2x_3 \le 5, \quad x_2 + 4x_3 + 5x_4 \le 3, \quad \textit{and} \quad x_1, x_2, x_3, x_4 \ge 0.$$

is given by the final simplex table given below,

Table 9·7

		$c_j \rightarrow$	10	3	6	5	0	0	0	
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{S_1}$	$\mathbf{S_2}$	$\mathbf{S_3}$	
x_2	3	56/27	0	1	−22/27	0	5/9	−5/27	−1/9	
x_1	10	5/3	1	0	2/3	0	0	1/3	0	
x_4	5	5/27	0	0	26/27	1	−1/9	1/27	2/9	
	$z = \mathbf{C_B X_B} = 643/27$		0	0	82/27	0	10/9	80/27	7/9	$\leftarrow \Delta_j$

Compute the limits for a_{11} so that the new solution remains optimal feasible solution.

Solution. Let us compute the limits for Δa_{11}. Here, $i = 1, j = 1, p = 2$ $\quad (\mathbf{a}_1 = \boldsymbol{\beta}_2)$

Now, we have

$$x_{B1}\beta_{21} - x_{B2}\beta_{11} = \frac{56}{27} \times 0 - \frac{5}{3} \times \frac{5}{9} = -\frac{25}{27} < 0$$

$$x_{B3}\beta_{21} - x_{B2}\beta_{31} = \frac{5}{27} \times 0 - \frac{5}{3} \times (-\frac{1}{9}) = \frac{5}{27} > 0.$$

Therefore, from (9·30), we get

$$-\frac{5}{27} \cdot \frac{27}{5} \le \Delta a_{11} \le -\frac{56}{27} \cdot \left(-\frac{27}{25}\right) \quad \text{or} \quad -1 \le \Delta a_{11} \le \frac{56}{25}. \qquad \text{...(A)}$$

Also, we have

$$\mathbf{C_B}\boldsymbol{\beta}_1 = 3\left(\frac{5}{9}\right) + 10 \times 0 + 5\left(-\frac{1}{9}\right) = \frac{10}{9}$$

$$\therefore \quad \beta_{21}\,\Delta_3 - x_{23}\,\mathbf{C_B}\boldsymbol{\beta}_1 = 0 - \frac{2}{3} \times \frac{10}{9} = -\frac{20}{27} < 0, \quad \beta_{21}\,\Delta_5 - x_{25}\,\mathbf{C_B}\boldsymbol{\beta}_1 = 0 - 0 = 0$$

$$\beta_{21}\,\Delta_6 - x_{26}\,\mathbf{C_B}\boldsymbol{\beta}_1 = 0 - \frac{1}{3} \cdot \frac{10}{9} = -\frac{10}{27} < 0, \quad \beta_{21}\,\Delta_7 - x_{27}\mathbf{C_B}\boldsymbol{\beta}_1 = 0 - 0 = 0.$$

Therefore, from (9·32), we compute

$$-\infty < \Delta a_{11} \le \min.\left[\frac{-82}{27} \times \frac{-27}{20}, \frac{-80}{27} \times \frac{-27}{10}\right] \quad \text{or} \quad -\infty < \Delta a_{11} \le 41/10. \qquad \text{...(B)}$$

Since (A) and (B) are to be satisfied simultaneously, we get $-1 \le \Delta a_{11} \le 56/25$.

EXAMINATION PROBLEMS

1. **Given the L.P.P. :** Max. $z = 10x_1 + 3x_2 + 6x_3 + 5x_4$, subject to the constraints :

   $x_1 + 2x_2 + x_4 \le 6, \quad 3x_1 + 2x_3 \le 5, \quad x_2 + 4x_3 + 5x_4 \le 3$ and $x_1, x_2, x_3, x_4 \ge 0$.

   (a) Determine an optimum solution to the problem.

   (b) If the element a_{11} is changed to $a_{11} + \Delta a_{11}$, determine the limits for discrete change Δa_{11} so as to maintain the optimality of the current optimum solution.

   (c) Determine the separate ranges for discrete change in a_{13}, a_{23} and a_{33} consistent with the optimality of the solution obtained in (a).

   [**Ans.** (a) $x_1 = 5/3$, $x_2 = 56/27$, $x_3 = 0$, $x_4 = 5/27$; max. $z = 643/27$ (b) $-1 \le \Delta a_{11} \le 56/25$
   (c) $-41/15 \le \Delta a_{13}$, $\infty < -41/40 \le \Delta a_{23} < \infty$, $-82/21 \ge \Delta a_{33} < \infty$.]

2. **Given the L.P.P. :** Max. $z = 3x_1 + 4x_2 + 2x_3 + 4x_4 + 2x_5 + 5x_6$, subject to the constraints :

   $x_1 - x_2 + x_3 + x_4 + 2x_6 = 10$, $2x_2 + 2x_4 + x_5 + 3x_6 = 6$, $x_1 + 4x_2 + 2x_3 + 5x_4 + x_5 + 9x_6 = 25$, and $x_j \ge 0$, $j = 1, 2, \ldots 6$.

   (a) Determine the optimum solution to the problem.

   (b) Discuss the effect of each of the following post-optimal discrete changes in the activity coefficients (taken one at a time) on the optimality of the solution obtained in (a) :

   (i) Change a_{24} to 1; (ii) change a_{25} to 2; and (iii) change a_{31} to -1.

   [**Ans.** (a) $x_1 = 1$, $x_2 = 0$, $x_3 = 9$, $x_4 = 3$, $x_5 = 6$, $x_6 = 0$, max $z = 33$
   (b) (i) $x_1 = 7$, $x_2 = 0$, $x_3 = 0$, $x_4 = 3$, $x_5 = 3$, $x_6 = 0$, max $= z = 39$
   (ii) Optimum solution is not affected]

9.8. CHANGES IN THE STRUCTURE OF L.P.P.

We shall now consider the post-optimal effects when some variable (s) or constraint(s) is/are added to or deleted from an L.P.P. after having obtained its optimal solution.

Here we shall discuss the post-optimality analysis for the following simplest type of structural changes only :

(a) *Addition of a new non-negative variable* (b) Deletion of an existing non-negative variable

(c) *Addition of a new constraint, and* (d) *Deletion of an existing constraint.*

However, analysis involving multiple additions or deletions can be easily handled by repeated application of the methods discussed for the simple cases. The first type (a) is the easiest to deal with.

9·8–1 Addition of New Variable

In this section, we want to discuss the necessary computation which have to be performed, if a new variable is to be introduced in a linear programming problem whose optimal solution is known. Here, we shall answer the question–'*Is there any way to obtain the new optimal solution without restarting the simplex method from the very beginning ?*'

Let us consider the L.P.P. : Max. $z = \mathbf{CX}$, *s.t.* $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \geq \mathbf{0}$ where $\mathbf{C}, \mathbf{X} \in R^n$, $\mathbf{b} \in R^m$ and $\mathbf{A}$ is an $m \times n$ activity matrix.

We assume that an optimal basic feasible solution $\mathbf{X_B}$ has been obtained. We now wish to discuss the effect on the optimality of this solution if some new non-negative variable x_{n+1}, having activity column $\mathbf{a}_{n+1}$ and corresponding cost coefficient c_{n+1}, is added to the given problem.

Obviously, it is only required to compute :

$$\mathbf{X}_{n+1} = \mathbf{B}^{-1}\,\mathbf{a}_{n+1} \text{ and } \Delta_{n+1} \equiv z_{n+1} - c_{n+1} = \mathbf{C_B}\,\mathbf{X}_{n+1} - c_{n+1},$$

where $\mathbf{B}$ is the current optimal basis and $\mathbf{C_B}$ is the associated cost vector.

Then, two possible cases are : (i) $\Delta_{n+1} \equiv z_{n+1} - c_{n+1} \geq 0$, (ii) $\Delta_{n+1} < 0$.

Obviously, in case (i), the current optimal basic solution ($\mathbf{X_B}$) remains optimal for the new problem. While in case (ii), the current optimal solution ($\mathbf{X_B}$) may be improved further by introducing $\mathbf{X}_{n+1}$ into the basis. The usual simplex routine can be applied, considering the optimal table of the original problem augmented by an $(n+1)^{th}$ column vector $\mathbf{X}_{n+1}$ as the starting table.

Addition of new variable is really just a special type of simultaneous changes in the objective function coefficients (c_j) and the coefficient a_{ij} of a corresponding non-basic variable. Consequently, the addition of a new variable can only affect the optimality of the problem.

This means that the new variable can enter the solution, if and only if, it improves the value of the objective function. Otherwise, the new variable becomes just like a non-basic variable (having zero value)

To explain this point, we discuss the following numerical example.

Example 9. *Consider to addition of a new variable x_5 to the following given L.P.P. (called the original primal) :*

$$\left.\begin{aligned} &\textit{Max. } z_x = 3x_1 + 5x_2, \textit{ subject to} \\ &x_1 \quad + x_3 \quad = 4 \\ &3x_1 + 2x_2 \quad + x_4 = 18 \\ &x_1, x_2, x_3, x_4 \geq 0 \end{aligned}\right\} \qquad \text{...(9·33)}$$

so that its revised form becomes :

$$\left.\begin{aligned} &\textit{Max. } z_x = 3x_1 + 5x_2 + 7x_5, \textit{ subject to} \\ &x_1 \quad + x_3 \quad + x_5 = 4 \\ &3x_1 + 2x_2 \quad + x_4 + 2x_5 = 18 \\ &x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}\right\} \qquad \text{...(9·34)}$$

Solution. By simplex method, we obtain the final table for optimal solution of problem (9·33) as follows :

Optimal Simplex Table 9·8

		$c_j \rightarrow$	3	5	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	
x_3	0	4	1	0	1	0	
x_2	5	9	3/2	1	0	1/2	
	$z = \mathbf{C_B X_B} = 45$		9/2	0	0	5/2	$\leftarrow \Delta_j$

The optimal solution is : $x_1 = 0$, $x_2 = 9$, $x_3 = 4$, $x_4 = 0$, with max. $z_x = 45$.

The optimal solution to the corresponding dual problem can be read from the above table as :

$$w_1 = 0 \, , \, w_2 = 5/2 \, , \, w_3 = 9/2 \, , \, w_4 = 0 \, , \text{ with } z_w = 45 \, .$$

We now return to the revised problem (9·34). The first question to be answered is whether $x_1 = 0$, $x_2 = 9$, $x_3 = 4$, $x_4 = 0$, $x_5 = 0$ (by appending $x_5 = 0$ to the previous optimal solution), will provide us the new optimal solution.

This question can easily be answered by refering to the dual problem. The complementary dual solution remains the same : $w_1 = 0$, $w_2 = 5/2$. The only change in the dual problem is the addition of new constraint, $w_1 + 2w_2 \geq 7$, which does not hold for this solution. Since the dual solution is not feasible, the corresponding primal solution is not optimal. So x_5 must enter the solution with positive value. To obtain the new optimal solution, we start by modifying the previous optimal table to include x_5 . The changes in the optimal table are obtained as below :

Table 9·9

		$c_j \rightarrow$	3	5	0	0	7	
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5} = \mathbf{B}^{-1} \mathbf{a_5}$	MIN $(\mathbf{X_B}/\mathbf{X_k})$
x_3	0	4	1	0	1	0	$\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1^* \\ 1 \end{pmatrix}$	$4/1 \leftarrow$
x_2	5	9	3/2	1	0	1/2		9/1
	$z = \mathbf{C_B X_B}$ = 45		9/2	0	0 $\downarrow$	5/2	$\Delta_5 = \mathbf{C_B X_5} - c_5$ $= (0, 5)\,(1, 1) - 7 = -2$	$\leftarrow \Delta_j$

We can further apply regular simplex method with $\mathbf{X_5}$ as the next entering basis vector.

By min. ratio rule (min $\mathbf{X_B}/\mathbf{X_5}$; $\mathbf{X_5} > 0$), we find that the key element is 1*. Consequently, the vector $\mathbf{X_3}$ must be removed from the basis and the new transformed Table 9·10 will be obtained as follows.

Table 9·10

		$c_j \rightarrow$	3	5	0	0	7	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	
x_5	7	4	1	0	1	0	1	
x_2	5	5	1/2	1	-1	1/2	0	
	$z = 53$		13/2	0	2	5/2	0	$\leftarrow \Delta_j$

$\Delta_1 = \mathbf{C_B X_1} - c_1 = (7, 5)\,(1, 1/2) - 3 = 13/2$, $\Delta_3 = \mathbf{C_B X_3} - c_3 = (7 , 5)\,(1 , -1) - 0 = 2$,

$\Delta_4 = \mathbf{C_B X_4} - c_4 = (7 , 5)\,(0 , 1/2) - 0 = 5/2$.

Hence, the new optimal solution will be $x_1 = 0$, $x_2 = 5$, $x_5 = 4$, with max. $z = 53$.

Example 10. *Let the optimum simplex table for a maximization problem (with all constraints of $\leq$ type) be*

	$c_j \rightarrow$		5	12	4	0	$-M$	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	
x_2	12	8/5	0	1	$-1/5$	2/5	$-1/5$	
x_1	5	9/5	1	0	7/5	1/5	2/5	
	$z = 141/5$		0	0	3/5	29/5	$M-2/5$	$\leftarrow \Delta_j$

where x_4 is slack and a_1 an artificial variable. Let a new variable $x_5 \geq 0$ be introduced in the problem with a cost 30 assigned to it in the objective function. Also suppose that the coefficients of x_5 in the two constraints are 5 and 7, respectively.

Discuss the effect of this addition of a variable on the optimality of the optimum solution to the given problem.

Solution. It is observed from the given table that $\mathbf{B}^{-1} = (\mathbf{X_4}, \mathbf{A_1}) \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix}$ and $\mathbf{a_5} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ is the activity vector associated with x_5.

Now,
$$\mathbf{X_5} = \mathbf{B}^{-1}\mathbf{a_5} = \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix}\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 19/5 \end{pmatrix}.$$

Since the cost vector associated with current $\mathbf{X_B}$ is $\mathbf{C_B} = (12, 5)$, therefore
$$\Delta_5 = \mathbf{C_B}\,\mathbf{X_5} - c_5 = (12, 5)\begin{pmatrix} 3/5 \\ 19/5 \end{pmatrix} - 30 = -19/5 < 0.$$

Also since $\Delta_5 < 0$, the optimality condition is violated as $\mathbf{X_5}$ enters the basis. The new simplex table now becomes :

	$c_j \rightarrow$		5	12	4	0	30	$-M$	
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{A_1}$	MIN. RATIO $\mathbf{X_B}/\mathbf{X_5}$
x_2	12	8/5	0	1	$-1/5$	2/5	3/5	$-1/5$	8/3
x_1	5	9/5	1	0	7/5	1/5	$\boxed{19/5}$	2/5	$9/19 \leftarrow$
	$z = 141/5$		0 $\downarrow$	0	3/5	29/5	$-19/5$ $\uparrow$	$M-2/5$	$\leftarrow \Delta_j$

Final Iteration. Remove $\mathbf{X_1}$ and introduce $\mathbf{X_5}$.

	$c_j \rightarrow$		5	12	4	0	30	$-M$
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{A_1}$
x_2	12	25/19	$-3/19$	1	$-8/19$	7/19	0	$-5/19$
x_5	30	9/19	5/19	0	7/19	1/19	1	2/19
	$z = 30$		1	0	2	6	0	$M \leftarrow \Delta_j$

Hence the new optimal basic feasible solution becomes :
$$x_1 = 0,\ x_2 = 25/19,\ x_3 = 0 \text{ and } x_5 = 9/19,\ \max z = 30.$$

Example 11. (a) *Discuss the effect of adding a new non-negative variable* $\mathbf{x_8}$ *in the L.P.P. of Example 6 (a), on the optimality of its optimal solution. It is given that the coefficient of* $\mathbf{x_8}$ *in the constraints of the problem are 2, 7, and 3 respectively; the cost component associated with* x_8 *being 5.*

(b) Explain the situation when we have $\mathbf{c_8} = \mathbf{10}$ *instead of 5.*

Solution. (a) The new L.P.P. becomes : Max. $z = 3x_1 + 4x_2 + x_3 + 7x_4 + 5x_8$, subject to the constraints :
$$8x_1 + 3x_2 + 4x_3 + x_4 + 2x_8 \leq 7$$
$$2x_1 + 6x_2 + x_3 + 5x_4 + 7x_8 \leq 3$$

$$x_1 + 4x_2 + 5x_3 + 2x_4 + 3x_8 \leq 8$$
$$x_j \geq 0, j = 1, 2, 3, 4, 8.$$

Instead of re-solving this new problem we wish to use the information contained in the current basic solution $\mathbf{x_B} = (16/19, 5/19, 126/19)$ and observe whether it still remains optimum for the new problem.

The activity vector associated with x_8 is $\mathbf{a_8}$ (2, 7, 3). From ***example 6 (a),*** we take

$$\mathbf{B}^{-1} = \begin{bmatrix} 5/38 & -1/38 & 0 \\ -1/19 & 8/38 & 0 \\ -1/38 & -15/38 & 1 \end{bmatrix}$$

and then calculate the entering vector $\mathbf{X_8} = \mathbf{B}^{-1}\mathbf{a_8} = [3/38, 26/19, 7/38]$

Since the cost vector associated with the current $\mathbf{x_B}$ is $\mathbf{C_B} = (3, 7, 0)$, we have

$$\Delta_8 = \mathbf{C_B X_8} - c_8 = (3, 7, 0)(3/38, 26/19, 7/38) - 5 = 183/38 > 0$$

Since $\Delta_8 > 0$, the optimality of the current solution $\mathbf{x_B}$ is not affected by the post-optimal addition of new variable x_8.

(b) If we have $c_8 = 10$ instead of 5, then we would have obtained $\Delta_8 = -7/38$ indicating that the optimality condition is violated. Then a new (improved) optimum soution can be obtained by using usual simplex method starting with the following augmented optimum simplex table of ***Example 6 (a)***:

		$c_j \rightarrow$	3	4	1	7	0	0	0	10	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{X_6}$	$\mathbf{X_7}$	$\mathbf{X_8}$	Min Ratio column
x_1	3	16/19	1	9/38	1/2	0	5/38	−1/38	0	3/38	32/3
x_4	7	5/19	0	21/19	0	1	−1/19	4/19	0	26/19	5/26 ←
x_7	0	126/19	0	59/38	9/2	0	−1/38	−15/38	1	7/38	252/7
	$z = 83/19$		0	169/38	1/2	0 ↓	1/38	53/38	0	−7/38 ↑	← Δ_j

It is easy to obtain the revised table by taking 26/19 as key element.

9·8-2 . Deletion of Existing Variable

Let the existing variable x_p be deleted from the given L.P.P. : max. $z = \mathbf{CX}$, subject to $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \geq \mathbf{0}$, where $\mathbf{C}$, $\mathbf{X} \in R^n$ and $\mathbf{b} \in R^m$.

We now wish to investigate the effect to this deletion on the optimality of the current optimal basic solution $\mathbf{X_B}$. Here two cases will arise : **Case 1 :** $x_p \notin \mathbf{X_B}$, **Case 2 :** $x_p \in \mathbf{X_B}$.

In the first case, the deletion of x_p would be a totally superfluous operation and thus we have no concern with this case (becasue, $x_p \notin \mathbf{X_B}$ does not change $\mathbf{X_B}$)

In the second case, let $x_p = x_{Br} \in \mathbf{X_B}$. There will be two different ways to tackle this situation.

(1) Make x_p $(= x_{Br})$ non-basic using the dual simplex entry criterion :
This will result in a new basic solution, that is, dual feasible but not necessarily primal feasible. Then, deleting the column of $\mathbf{X_p}$ from the simplex table, we can re-optimize using the dual simplex method.

(2) Assign a largest negative cost (–M) to x_p and then apply the standard (primal) simplex method, taking the modified current optimal simplex table as the starting table to obtain a new optimum solution. Here we shall explain the *method (2)* only.

Example 12. *Let us once again consider the Example 6(a). Let the variable x_4 be deleted from the L.P.P. of the example 6 (a) obtain an optimum solution to the resulting L.P.P.*

Solution. The variable to be deleted is a basic variable. So assign a cost –M to x_4 and consider the optimum simplex table as the initial simplex table for the L.P. problem.

Starting Simplex Table

BASIC VAR.	C_B	X_B	X_1 (3)	X_2 (4)	X_3 (1)	X_4 (−M)	X_5 (0)	X_6 (0)	X_7 (0)	MIN. RATIO (X_B/X_2)
x_1	3	16/19	1	9/38	1/2	0	5/38	− 1/38	0	32/9
x_4	−M	5/19	0	$\boxed{21/19}$ ←	0	1	1/19	−4/19	0	5/21 ←
x_7	0	126/19	0	59/38	9/2	0	− 1/38	− 15/38	1	258/59
	$z = \frac{(48-5M)}{19}$		0	$-\frac{(42M+125)}{38}$ ↑	1/2	0 ↓	$\frac{(2M+15)}{38}$	$-\frac{(8M+3)}{38}$	0	← Δ_j

First Iteration. Insert X_2 and remove X_4.

BASIC VAR.	C_B	X_B	X_1 (3)	X_2 (4)	X_3 (1)	X_4 (—M)	X_5 (0)	X_6 (0)	X_7 (0)	
x_1	3	11/14	1	0	1/2	×	1/7	− 1/14	0	
x_2	4	5/21	0	1	0.	×	− 1/21	4/21	0	
x_7	0	5123/58	0	0	9/2	×	1/21	− 29/42	1	
	$z = 139/42$		0	0	1/2	×	5/21	23/42	0	← Δ_j

Thus, the new optimum solution becomes : $x_1 = 11/14$, $x_2 = 5/21$, $x_3 = 0$, max $z = 139/42$.

9·8-3 . Addition of New Constraint

Let us consider the L.P.P. : Max. $z = \mathbf{CX}$, s.t. $\mathbf{AX} = \mathbf{b}$, $\mathbf{X} \geq \mathbf{0}$, where $\mathbf{C}, \mathbf{X} \in R^n$, $\mathbf{b} \in R^m$, and $\mathbf{A}$ is an $m \times n$ real activity matrix.

First re-arrange the columns of $\mathbf{A}$ so that the first m column vectors of $\mathbf{A}$ form a basis set. Originally we have m constraints in the problem.

To proceed further, we assume that one more constraint : $\mathbf{a}^T \mathbf{X} \leq b_{m+1}$, where $\mathbf{a} \in R^n$ and b_{m+1} is scalar constant, be added post-optimality to the given L.P.P.

Now our aim is to study the effect of this modification on the optimality of the current optimum basic feasible solution $\mathbf{X_B}$ of the given problem. Then two cases are of our interest :

(1) $\mathbf{X_B}$ satisfies the new set of constraints. (2) $\mathbf{X_B}$ does not satisfy the new set of constraints.

Case 1. If case **1** occurs, $\mathbf{X_B}$ satisfies, that $\mathbf{X_B}$ remains optimum for the post-optimal L.P.P. It follows from the fact that a redundent (extra) constraint cannot enlarge the feasible space of an L.P.P. However, it can either reduce the feasible space or leave it unchanged. Thus, $\mathbf{X_B}$ will remain optimum for the new problem, because if $\mathbf{X_B}$ remains feasible it will remain optimum.

Case 2. If case (2) occurs, we have to search for some other optimum solution. For this, we convert a new constraint to standard form by introducing a slack variable x_s . Thus the resulting L.P.P. has the following $(m + 1)$ constraints:

$$\mathbf{AX} = \mathbf{b} \qquad \text{(original } m\text{-constraints)}$$
$$\mathbf{a}^T\mathbf{X} + x_s = b_{m+1} , \qquad (m+1)th \text{ constraint.}$$

Where $\mathbf{X}$ and x_s are required to be non-negative as usual. Then we have to show that $\mathbf{X_B}^* = \begin{bmatrix} \mathbf{X_B} \\ x_s \end{bmatrix}$ is a basic feasible solution to the new L.P.P. To prove this, we collect the activity columns corresponding to x_{B1} , x_{B2} , ... , x_{Bm} , x_s into an $(m + 1) \times (m + 1)$ basis matrix :

$$\mathbf{B}^* = \begin{pmatrix} \mathbf{B} & 0 \\ \mathbf{u} & 1 \end{pmatrix},$$

where $\mathbf{u} = (a_{m+1,1}$, $a_{m+1,2}$, ... ; $a_{m+1,n})$, where $a_{m+1,j}$ being the jth component of $\mathbf{u}$. Obviously, $\mathbf{B}^*$ will form a basis matrix since it has an inverse

$$(\mathbf{B}^*)^{-1} = \begin{pmatrix} \mathbf{B}^{-1} & 0 \\ -\mathbf{u}\,\mathbf{B}^{-1} & 1 \end{pmatrix}$$

It can be verified that $(\mathbf{B}^*)\,(\mathbf{B}^*)^{-1} = \mathbf{I}_{m+1}$.

Now, with the help of above inverse of new basis, we can easily compute all the required values to complete the simplex table for the new problem.

Computation of $\mathbf{x_B}^*$. The new values of basic variables are given by,

$$\mathbf{x_B}^* = \begin{bmatrix} \mathbf{x_B} \\ \mathbf{x_s} \end{bmatrix} = (\mathbf{B}^*)^{-1}\,\mathbf{b}^* = \begin{pmatrix} \mathbf{B}^{-1} & 0 \\ -\mathbf{u}\,\mathbf{B}^{-1} & 1 \end{pmatrix}\begin{pmatrix} \mathbf{b} \\ b_{m+1} \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ b_{m+1} - \mathbf{u}\,\mathbf{B}^{-1}\mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{x_B} \\ b_{m+1} - \mathbf{u}\,\mathbf{x_B} \end{pmatrix}$$

where $\mathbf{b}^*$ denotes the new requirement vector.

Computation of $\mathbf{x_j}^*$. Now for the new problem, each column of activity matrix becomes $\mathbf{a_j}^* = (a_j, a_{m+1,j})$ and, therefore, $\mathbf{x_j}^*$ is computed by,

$$\mathbf{x_j}^* = (\mathbf{B}^*)^{-1}\,\mathbf{a_j}^* = \begin{pmatrix} \mathbf{B}^{-1} & 0 \\ -\mathbf{u}\,\mathbf{B}^{-1} & 1 \end{pmatrix}\begin{pmatrix} \mathbf{a_j} \\ a_{m+1,j} \end{pmatrix} = \begin{pmatrix} \mathbf{x_j} \\ a_{m+1,j} - \mathbf{u}\,\mathbf{B}^{-1}\mathbf{a_j} \end{pmatrix}.$$

Computation of net evaluation Δ_j^*. The net evaluation for any non-basic variable x_j is given by

$$\Delta_j^* = \mathbf{C_B}^*\mathbf{x_j}^* - c_j = (\mathbf{C_B}, 0)\begin{pmatrix} \mathbf{x_j} \\ a_{m+1,j} - \mathbf{u}\,\mathbf{B}^{-1}\mathbf{a_j} \end{pmatrix} - c_j = \mathbf{C_B}\,\mathbf{x_j} - c_j = \Delta_j,$$

which proves to be the same as before.

Computation of z^*. We know, $z^* = (\mathbf{C_B}, \mathbf{0})\,(\mathbf{x_B}, \mathbf{x_s}) = \mathbf{C_B}\,\mathbf{x_B} = z$.

From above computations, we observe that there is no change in the entries of optimum simplex table of the given problem. We only need to add a new row for the $(m+1)th$ basic variable x_j whose value and $\mathbf{x_j}$-components are given by above formulae for $\mathbf{x_B}^*$ and $\mathbf{x_j}^*$, respectively. Then, we can easily solve the new post-optimal problem by using *dual simplex method.*

Remarks :

1. Since the sign of b_{m+1} is not specified in the additional constraint, so a constraint of opposite inequality form can be multiplied throughout by -1 and then added to the given problem in the like manner.
2. If the last column in $\mathbf{B}^*$ occurs due to an artificial variable (instead of slack variable), we assign a cost $-M$ to that variable and then apply the usual simplex method to obtain an optimum solution. It is worth-noting that the net evaluations $z_j - c_j$ do not remain unchanged in this situation.
3. Whenever more than one constraints are added to the given problem and all are not satisfied by the original optimum solution, then it is advisable to solve the problem anew.

Working Rule. ***We suppose that, after obtaining the optimal solution, it is decided that a new constraint should be introduced to the given problem. This newly introduced constraint can affect the feasibility of the present optimal solution only if it is active. Consequently, the first step would be to judge whether the new constraint is satisfied by the present optimal solution or not. If the constraint is satisfied, the newly added constraint is redundant and the optimal solution remains unchanged. If the constraint is not satisfied, the new constraint must be introduced to the system of constraints as shown below by a numerical example. We add a slack variable and also add the resulting equation to the final set of equations. Let the new slack variable be the basic variable for the new equation and we algebraically eliminate any other basic variables that appear in the equation. Although the conditions of optimality will still be satisfied, but the solution will be infeasible since the new slack variable will be negative. In fact, to start with infeasible optimal solution and to progress towards feasible optimal solution is the situation where the dual simplex method should be used.***

To illustrate the procedure we discuss the following example.

Example 13. (a) *Discuss the effect to addition of new constraint $x_2 \le 10$ on the optimality of the optimal solution to the L.P.P. :*

$$\textit{Max. } z = 3x_1 + 5x_2\,,\ \textit{subject to } x_1 \le 4\,,\ 3x_1 + 2x_2 \le 18\,;\ x_2\,, x_2 \ge 0\,.$$

(b) *If, instead, the constraint $x_2 \le 6$ is added to the problem in (a) above, discuss the post-optimality analysis.*

Solution. (a) After introducing the slack variables x_3 and x_4 the given problem becomes :

$$\text{Max.}\quad z = 3x_1 + 5x_2 + 0x_3 + 0x_4,\ \text{subject to}$$

$$x_1 \quad + x_3 \quad = 4$$
$$3x_1 + 2x_2 \quad + x_4 = 18$$
$$x_1, x_2, x_3, x_4 \quad \geq 0 .$$

Applying usual simplex method we have obtained the optimal ***simplex table* 9·8**. The optimal solution thus obtained is : $x_1 = 0$, $x_2 = 9$, $x_3 = 4$, $x_4 = 0$.

When the new constraint $x_2 \leq 10$ is added to the given problem, we observe that this constraint is satisfied by the current optimal solution and it is redundant. That is, there is no effect of adding the new constraint $x_2 \leq 10$.

(b) Now, suppose the constraint, $x_2 \leq 6$, is added. It is observed that it is not satisfied by the current optimal solution. Hence the current optimal solution : ($x_1 = 0$, $x_2 = 9$, $x_3 = 4$, $x_4 = 0$) is no longer feasible. Thus in order to clear infeasibility, the new equation $x_2 + x_5 = 6$ must be added to the optimal tableau. Here x_5 is a slack variable. To do so, we need to eliminate the basic variable x_2 from the new equation. This is done by subtracting the proper multiple of given equation from the new equation, as follows :

$$0x_1 + 1x_2 + 0x_3 + 0x_4 + 1x_5 = 6 \qquad \textit{(new equation)}$$

$$\frac{3}{2}x_1 + 1x_2 + 0x_3 + \frac{1}{2}x_4 + 0x_5 = 9 \quad \textit{(after dividing the given equation by 2)}$$

On subtracting the second equation from first, we get

$$-\frac{3}{2}x_1 + 0x_2 + 0x_3 - \frac{1}{2}x_4 + x_5 = -3 .$$

This modified equation can be added as a third row in the optimal table 9·8 to give the following new table.

Table 9·11

		$c_j \rightarrow$	3	5	0	0	0	
BSIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$ (β_2)	$\mathbf{X_3}$ (β_1)	$\mathbf{X_4}$	$\mathbf{X_5}$ (β_3)	
x_3	0	4	1	0	1	0	0	
x_2	5	9	3/2	1	0	1/2	0	
x_5	0	$\rightarrow$ − 3	$\boxed{-3/2}$	0	0	− 1/2	1$\rightarrow$	
	$z = \mathbf{C_B X_B} = 45$		9/2 $\uparrow$	0	0	5/2	0 $\downarrow$	$\leftarrow \Delta_j$

$$\Delta_1 = \mathbf{C_B X_1} - c_1 = (0, 5, 0)\,(1, 3/2, -3/2) - 3 = 9/2$$
$$\Delta_4 = \mathbf{C_B X_4} - c_4 = (0, 5, 0)\,(0, \tfrac{1}{2}, -\tfrac{1}{2}) - 0 = 5/2 .$$

The resulting optimal solution becomes : $x_1 = 0$, $x_2 = 9$, $x_3 = 4$, $x_4 = 0$, $x_5 = -3$, which is infeasible because x_5 violates the non-negativity restrictions. However, applying the *dual simplex method,* starting with this table, soon gives the desired feasible optimal solution as follows :

To determine the leaving vector :

Since $x_{Br} = \min\,[x_{Bi}, x_{Bi} < 0] = \min\,[-3] = -3 = x_{B3}$.

Therefore, $r = 3$, which indicates that key row is the third one. So we must remove β_3, *i.e.* $\mathbf{X_5}$.

To determine the entering vector $\mathbf{a_k}$:

$$\frac{\Delta_k}{x_{3k}} = \max\left[\frac{\Delta_1}{x_{31}}, \frac{\Delta_4}{x_{34}} \text{ for } x_{31} < 0, x_{34} < 0\right] = \max.\left[\frac{9/2}{-3/2}, \frac{5/2}{-1/2}\right] = \frac{9}{-3} = \frac{\Delta_1}{x_{31}} .$$

Therefore, $k = 1$, which indicates that key column is the first one.

So we must enter the vector $\mathbf{a_1}$ corresponding to variable x_1.

Therefore, the key element is found to be [− 3/2] which is the intersection of third row and first column. We now get the following transformed *Table* **9·12**.

Table 9·12

		$c_j \rightarrow$	3	5	0	0	0	
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	
x_3	0	2	0	0	1	$-1/3$	$2/3$	
x_2	5	6	0	1	0	0	1	
x_1	3	2	1	0	0	$1/3$	$-2/3$	
	$z_x = 36$		0	0	0	1	3	$\leftarrow \Delta_j$

$$\Delta_4 = \mathbf{C_B X_4} - c_4 = (0\,,5\,,3)\,(-\tfrac{1}{3}\,,0\,,\tfrac{1}{3}) - 0 = 1,\ \Delta_5 = \mathbf{C_B X_5} - c_5 = (0,5\,,3)\,(\tfrac{2}{3}\,,1\,,-\tfrac{2}{3}) - 0 = 3\,.$$

Thus, the new optimal feasible solution will be read from above table as $x_1 = 2\,, x_2 = 6\,, x_3 = 2$, with max $z = 36$.

Example 14. *Consider the following table which presents optimal solution to some linear programming problem.*

BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{X_6}$	$\mathbf{X_7}$	$\mathbf{X_8}$	
x_1	2	3	1	0	0	-1	0	0·5	0·2	-1	
x_2	4	1	0	1	0	2	1	-1	-1	0·5	
x_3	1	7	0	0	1	-1	-1	5	$-0·3$	2	
	$z = 17$		0	0	0	2	0	2	0·1	2	$\leftarrow \Delta_j$

If the additional constraint $2x_1 + 3x_2 - x_3 + 2x_4 + 4x_5 \leq 5$ *were annexed to the system, would there be any change in the optimal solution ? Justify your answer.*

Solution. It is observed that $\mathbf{X_B} = [3\ \ 1\ \ 7]$ with $\mathbf{B} = \mathbf{I_3}$. After the addition of the given additional constraint to the problem, the new basis matrix becomes :

$$\mathbf{B}^* = \begin{pmatrix} \mathbf{I_3} & 0 \\ \mathbf{u} & 1 \end{pmatrix},\ \text{where } \mathbf{u} = (2 \quad 3 \quad -1)\,.$$

The optimum solution to the adjusted problem is then given by

$$\mathbf{X_B}^* = \begin{bmatrix} \mathbf{X_B} \\ -\mathbf{u}\mathbf{X_B} + 5 \end{bmatrix} = \begin{pmatrix} \mathbf{X_B} \\ 3 \end{pmatrix} = (3 \quad 1 \quad 7 \quad 3)\,.$$

Thus $\mathbf{X_B}^*$ is an optimum basic feasible solution to the new problem.

9·8-4 Deletion of Existing Constraint.

Now we shall consider the case when one of the existing constraints is deleted post-optimally from the problem. There are two possible cases :

(1) The constraint in question is a *binding* on the optimum solution $\mathbf{X_B}$.

(2) The constraint in question is not a *binding* on the optimum solution $\mathbf{X_B}$.

Obviously, case 1 gives rise to a post-optimality problem. Because, the deletion of a non-binding constraint can only enlarge the feasible region, and all new solutions are inferior to $\mathbf{X_B}$. This statement can be easily verified by graphical method. Also, if the constraint in question has a positive valued slack or surplus variable in the optimum solution $\mathbf{X_B}$, then the constraint cannot be a binding on $\mathbf{X_B}$ and hence $\mathbf{X_B}$ must be an optimum solution to the new problem.

To consider case 1, the earliest way to proceed is *via* the addition of one or two new variables. Let the constraint in question be

$$a_{i1}\,x_1 + a_{i2}\,x_2 + \ldots + a_{in}\,x_n = b_i$$

which is the ith constraint of the given problem. Now adding to it a slack and a surplus variable, as follows :

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + x_{n+1} - x_{n+2} = b_i\,,$$

where $x_{n+j} \geq 0\,, j = 1\,,2$.

This implies that we are allowing the L.H.S. of the constraint in question to be > or < b_i as well. For example, $(a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n) < b_i$ is quite feasible because the slack can be accomodated by a positive value of x_{n+2}. Thus, by introducing x_{n+1} and x_{n+2} we have, in fact, deleted the constraint in question. The procedure of adding new variables to an L.P.P. has already been discussed earlier.

9.9. MORE ILLUSTRATIVE EXAMPLES

Example 15. *Consider the LP problem : Max.* $z = 3x_1 + 4x_2 + x_3 + 7x_4$, *subject to the constraints :*

$8x_1 + 3x_2 + 4x_3 + x_4 \le 7$, $2x_1 + 6x_2 + x_3 + 5x_4 \le 3$, $x_1 + 4x_2 + 5x_3 + 2x_4 \le 8$, *and* $x_1, x_2, x_3, x_4 \ge 0$.

(a) *Determine the optimal solution.*

(b) *Discuss the effect of discrete changes in* b_i, *where* b_i $(i = 1, 2, 3)$ *are the constants on the right hand side.*

(c) *Discuss the effect of discrete changes in* c_j *on the optimality of the optimum basic feasible solution.*

(d) *Discuss the effect of discrete changes in the activity coefficients* a_{ij} *of A on the current optimum basic feasible solution.*

(e) *Let a linear constraint* $2x_1 + 3x_2 + x_3 + 5x_4 \le 4$ *be added to the constraints of the problem. Check whether there is any change in the optimum solution of the original problem.*

Also discuss the case when the upper limit of the above constraint is reduced to 2.

Solution. Introducing the slack variables $x_5 \ge 0$, $x_6 \ge 0$, and $x_7 \ge 0$, and then solving the problem by simplex method, the following simplex table is obtained :

Optimal Table 9.13

	$c_j \rightarrow$		3	4	1	7	0	0	0	
								B^{-1}		
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	X_5 (β_1)	X_6 (β_2)	X_7 (β_3)	
x_1	3	16/19	1	9/38	1/2	0	5/38	−1/38	0	
x_4	7	5/19	0	21/19	0	1	−1/19	4/19	0	
x_7	0	126/19	0	59/38	1/2	0	−1/38	−15/38	1	
	$z = 83/19$		0	169/38	1/2	0	1/38	53/38	0	$\leftarrow \Delta_j$

(*a*) The optimal solution to the given problem can be read off from the optimum table :

$$x_1 = 16/19,\ x_2 = 0,\ x_3 = 0,\ x_4 = 5/19,\ \max z = 83/19.$$

(*b*) Using the relationship

$$\underset{\beta_{il} > 0}{\underset{i}{\text{Max}}} \left[\frac{-x_{Bi}}{\beta_{il}}\right] \le \Delta b_l \le \underset{\beta_{il} < 0}{\underset{i}{\text{Min}}} \left[\frac{-x_{Bi}}{\beta_{il}}\right]$$

the individual effects of changes in b_1, b_2, b_3, such that optimality of new basic solution is not violated, are given by

(*i*) $\dfrac{-16/19}{5/38} \le \Delta b_1 \le \min\left[\dfrac{-5/19}{-1/19}, \dfrac{-126/19}{-1/38}\right]$ or $-32/5 \le \Delta b_1 \le 5$.

(*ii*) $\dfrac{-5/19}{4/19} \le \Delta b_2 \le \min\left[\dfrac{-16/19}{-1/38}, \dfrac{-126/19}{-15/38}\right]$ or $-5/4 \le \Delta b_2 \le 84/5$.

(*iii*) $\dfrac{-126/19}{1} \le \Delta b_3$ or $\dfrac{-126}{19} \le \Delta b_3$.

(*c*) The cost vector is $\mathbf{C} = (c_1, c_2, c_3, c_4, c_5, c_6, c_7)$ and the basic cost vector as $\mathbf{C_B} = (c_1, c_4, c_7)$. Then the changes in c_j's can be classified in two ways :

(1) changes in c_j when $j = 1, 4, 7$, (2) changes in c_j when $j = 2, 3, 5, 6$.

For (1) an optimal basic feasible solution will maintain its optimality if the change Δc_r in c_r satisfies the relationship :

$$\underset{x_{rj}>0}{\underset{j}{\text{Max}}}\left(\frac{-\Delta_j}{x_{rj}}\right) \le \Delta c_{Br} \le \underset{x_{rj}<0}{\underset{j}{\text{Min}}}\left(\frac{-\Delta_j}{x_{rj}}\right)$$

Thus the limits for changes in c_r's are given by

(*i*) $\max\left[\frac{-169/38}{9/38}, \frac{-1/2}{1/2}, \frac{-1/38}{5/38}\right] \le \Delta c_1 \le \frac{-53/38}{-1/38}$ or $-1/5 \le \Delta c_1 \le 53$.

(*ii*) $\max\left[\frac{-169/38}{21/19}, \frac{53/38}{8/38}\right] \le \Delta c_4 \le \frac{-1/38}{-1/19}$ or $-169/42 \le \Delta c_4 \le \frac{1}{2}$.

(*iii*) $\max\left[\frac{-169/38}{59/38}, \frac{-1/2}{9/2}\right] \le \Delta c_7 \le \min\left[\frac{-1/38}{-1/38}, \frac{-53/38}{-15/38}\right]$ or $-1/9 \le \Delta c_7 \le 1$.

For (2), the change Δc_r in c_r must satisfy the upper limit $\Delta c_r \le z_r - c_r$ in order to maintain the optimality of the optimum basic feasible solution.

Thus, $\Delta c_2 \le 169/38$, $\Delta c_3 \le \frac{1}{2}$, $\Delta c_5 \le 1/38$, $\Delta c_6 \le 53/38$.

(*d*) Compute $\mathbf{c_B}\beta_1 = (3\ \ 7\ \ 0)\,(5/38, -1/19, -1/38) = 1/38$,

$\mathbf{c_B}\beta_2 = (3\ \ 7\ \ 0)\,(-1/38, 8/38, -15/38) = 53/38$,

$\mathbf{c_B}\beta_3 = (3\ \ 7\ \ 0)\,(0\ \ 0\ \ 1) = 0$, where β_1, β_2, β_3 are the column vectors of $\mathbf{B}^{-1}$.

Let the element a_{ij} of $\mathbf{A}$ be changed to $a_{ij}^* = a_{ij} + \Delta a_{ij}$.

Case 1. a_{ij} is not in the basis B.

In the optimal table, $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_5$ **and** $\mathbf{x}_6$ are not in the basis, therefore the ranges for discrete changes in the coefficient a_{ij} corresponding to these non-basic vectors are given by

$(-169/38)/(1/38) \le \Delta a_{12} \Rightarrow \Delta a_{12} \ge -169$

$(-169/38)/(53/38) \le \Delta a_{22} \Rightarrow \Delta a_{22} \ge -169/53$

$(-169/38)/0 \le \Delta a_{32} < \infty \Rightarrow -\infty < \Delta a_{32} < \infty$

$(-\frac{1}{2})/(1/38) \le \Delta a_{13} \Rightarrow \Delta a_{13} \ge -19$

$(-\frac{1}{2})/(53/38) \le \Delta a_{23} \Rightarrow \Delta a_{23} \ge -19/53$

$(-\frac{1}{2})/0 \le \Delta a_{33} < \infty \Rightarrow -\infty < \Delta a_{33} < \infty$

$(-1/38)/(1/38) \le \Delta a_{15} \Rightarrow \Delta a_{15} \ge -1$

$(-1/38)/(53/38) \le \Delta a_{25} \Rightarrow \Delta a_{25} > -1/53$

$(-1/38)/0 \le \Delta a_{35} \le \infty \Rightarrow -\infty < \Delta a_{35} < \infty$

$(-53/38)/(1/38) \le \Delta a_{16} \Rightarrow \Delta a_{16} \ge -53$

$(-53/38)/(53/38) \le \Delta a_{26} \Rightarrow \Delta a_{26} \ge -1$

$-\infty < \Delta a_{36} < \infty$.

Case 2. a_{ij} in the basis B.

Since $\mathbf{x}_1$, $\mathbf{x}_4$ and $\mathbf{x}_7$ are in the basis, therefore any discrete change in $a_{ij} \in \mathbf{B}$ may affect the feasibility as well as the optimality of the original optimum basic feasible solution $\mathbf{x_B}$.

Consider the discrete changes in a_{ij} belonging to $\mathbf{x}_4 = \beta_2$.

Compute, $\beta_{22}(z_2 - c_2) - x_{22}\,\mathbf{c_B}\,\beta_2 = \frac{8}{38}\left(\frac{169}{38}\right) - \frac{21}{19}\left(\frac{53}{38}\right) = -\frac{23}{38}$

$\beta_{22}(z_3 - c_3) - x_{23}\,\mathbf{c_B}\,\beta_2 = \frac{8}{38}\left(\frac{1}{2}\right) - 0\left(\frac{53}{38}\right) = \frac{2}{19}$

$\beta_{22}(z_5 - c_5) - x_{25}\,\mathbf{c_B}\,\beta_2 = \frac{8}{38}\left(\frac{1}{38}\right) - \left(-\frac{1}{19}\right)\left(\frac{53}{38}\right) = \frac{3}{38}$

$\beta_{22}(z_6 - c_6) - x_{26}\,\mathbf{c_B}\,\beta_2 = \frac{8}{38}\left(\frac{53}{38}\right) - \frac{8}{38}\left(\frac{53}{38}\right) = 0$

Therefore, the range for the discrete change in a_{24} is given by :

$$\max\left[\frac{-1/2}{2/19}, \frac{-1/38}{3/38}\right] \le \Delta a_{24} \le \min\left[\frac{-169/38}{23/38}\right] \quad \text{or} \quad -\tfrac{1}{3} \le \Delta a_{24} \le 169/23. \qquad \text{...(A)}$$

Further, since

$$x_{B1}\,B_{24} - x_{B2}\,B_{21} = x_{B1}\,\beta_{22} - x_{B2}\,\beta_{12} = \frac{16}{19}\left(\frac{8}{38}\right) - \frac{5}{19}\left(-\frac{1}{38}\right) = \frac{7}{38}$$

$$x_{B3}\,B_{24} - x_{B2}\,B_{22} = x_{B3}\,\beta_{22} - x_{B2}\,\beta_{32} = \frac{126}{19}\left(\frac{8}{38}\right) - \frac{5}{19}\left(-\frac{15}{38}\right) = \frac{3}{2}.$$

Therefore, the range for the discrete change in the element a_{24} in order to maintain the feasibility is given by

$$\max\left[\frac{-16/19}{7/38}, \frac{-126/19}{3/2}\right] \le \Delta a_{24} \qquad \text{or} \qquad -84/19 \le \Delta a_{24}. \qquad \text{...(B)}$$

Since (A) and (B) are to be satisfied simultaneously, $-\frac{1}{3} \le a_{24} \le 169/23$.

Similarly, the ranges for the discrete changes in the elements a_{14}, a_{34}, etc. can also be determined.

(e) Including the additional constraint to the problem, the new basis matrix becomes

$$\mathbf{B}^* = \begin{pmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{u} & \mathbf{1} \end{pmatrix} \text{where } \mathbf{u} = (2\ \ 5\ \ 0),$$

because a slack variable has been introduced in the given additional constraint. Thus the optimum solution to the new L.P. problem is given by

$$\mathbf{x_B}^* = \begin{bmatrix} \mathbf{x_B} \\ -\mathbf{u}\mathbf{x_B} + 4 \end{bmatrix} = \left[\frac{16}{19}, \frac{5}{19}, \frac{126}{19}, 1\right]$$

which is feasible also.

The value of the objective function remains unchanged, since zero cost is assigned to slack variables.

Let us now discuss the case when the additional constraint to be included is : $2x_1 + 3x_2 + x_3 + 5x_4 \le 2$.

Since $\mathbf{x}_1$, $\mathbf{x}_4$ and $\mathbf{x}_7$ are in the basis, corresponding coefficients in the additional constraint must vanish. This may be achieved by using the appropriate row operation and thus yielding the following table.

		$c_j \rightarrow$	3	4	1	7	0	0	0	0	
BASIS VAR.	$\mathbf{c_B}$	$\mathbf{x_B}$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_8$	
x_1	3	16/19	1	9/38	1/2	0	5/38	− 1/38	0	0	
x_4	7	5/19	0	21/19	0	1	− 1/19	4/19	0	0	
x_7	0	126/19	0	59/38	1/2	0	− 1/38	− 15/38	1	0	
x_8	0	− 1	0	− 3	0	0	0	$\boxed{-1}$	0	1	
	$z = 83/19$		0	169/38	1/2	0	1/38	53/38 ↑	0	0 ↓	$\leftarrow \Delta_j$

It is observed that a basic solution to the new problem becomes : $\mathbf{x}_\mathbf{B}^* = [16/19, 5/19, 126/19, -1]$ which is not feasible. Therefore, dual simplex method must be applied to obtain the optimum basic feasible solution to the new problem.

Now apply *dual simplex method* bacause $\mathbf{x_B}$ is not feasible here.

First Iteration. Remove $\mathbf{x}_8$ and insert $\mathbf{x}_6$.

		$c_j \rightarrow$	3	4	1	7	0	0	0	0	
BASIC VAR.	$\mathbf{c_B}$	$\mathbf{x_B}$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_8$	
x_1	3	33/38	1	6/19	1/2	0	5/38	0	0	− 1/38	
x_4	7	1/19	0	9/19	0	1	− 1/19	0	0	4/19	
x_7	0	267/38	0	52/19	1/2	0	− 1/38	0	1	− 15/38	
x_6	0	1	0	3	0	0	0	1	0	− 1	
	$z = 113/38$		0	5/19	1/12	0	1/38	0	0	53/38	$\leftarrow \Delta_j$

Thus, addition of new constraint has increased the optimum value of the objective function from 83/19 to 113/38.

SELF-EXAMINATION QUESTIONS

1. (a) Write an explanatory note on sensitivity analysis.**[AIMS (Bang.) MBA 2002; Delhi B.Sc. (Maths.) 90; Meerut 90]**

   (b) What do you understand by th term sensitivity analysis ? Discuss briefly :

   (i) Variation of the c_j. (ii) Variation of the b_j.

2. Given that the problem : Max $z = \mathbf{cx}$ such that $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ has an optimum solution, can one obtain a linear programming problem which has an unbounded solution changing **b** alone ?.
3. Consider the following L.P.P. :

   Max. $z = \mathbf{cx}$, subject to $\mathbf{Ax} = \mathbf{b}$, and $\mathbf{x} \geq 0$, where $\mathbf{c}, \mathbf{x} \in R^n$, $\mathbf{b} \in R^m$ and A is any $m \times n$ real matrix. Determine how much can be components of the cost vector **c** be changed without affecting the optimal solution of the L.P.P.
4. What do you mean by 'sensitivity analysis' ? Discuss sensitivity analysis with respect to :
   (a) Change in the constraint matrix ; and (b) Addition of a new constraint.
5. Find the limits of variation of a_{ik} so that the optimal feasible solution of $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq 0$, max. $z = \mathbf{cx}$ remains optimal feasible solution, when $\mathbf{a}_k$ is not a vector of the optimal base.
6. Describe the role of duality for sensitivity analysis of a linear programming problem.
7. Explain with suitable examples the basic philosophy behind sensitivity analysis.

EXAMINATION PROBLEMS

1. Consider the LP problem : Max. $z = 5x_1 + 12x_2 + 4x_3$, subject to the constraints :
$$x_1 + 2x_2 + x_3 \leq 5, \quad 2x_1 - x_2 + 3x_3 = 2, \quad \text{and} \quad x_1, x_2, x_3 \geq 0.$$
The optimum solution to this problem is contained in the following table.

BASIC VAR.	X_B	X_1	X_2	X_3	X_4	A_1	
x_2	8/5	0	1	– 1/5	2/5	– 1/5	
x_1	9/5	1	0	7/5	1/5	2/5	
	$z = 28\frac{1}{5}$	0	0	3/5	29/5	M – 2/5	$\leftarrow \Delta_j$

For each of the discrete parameters listed below, make the necessary corrections in the optimum table and solve the resulting problem :

(a) change **b** to (3 10) from (5, 2), (b) change **c** to (4, 10, 4) from (5, 12, 4), (c) change $\mathbf{a}_3$ to $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ from $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

2. Given the LP problem : Max $z = 4x_1 + 3x_2 + 4x_3 + 6x_4$ subject to the constraints :
$$x_1 + 2x_2 + 2x_3 + 4x_4 \leq 8, \quad 2x_1 + 2x_3 + x_4 \leq 6, \quad 3x_1 + 3x_2 + x_3 + x_4 \leq 8_2 \quad \text{and} \quad x_1, x_2, x_3, x_4 \geq 0.$$
   (a) Determine the separate ranges for discrete changes in a_{22}, a_{32} and a_{23} consistent with the optimum solution of the given LP problem.
   (b) If a_{22} is changed to $a_{22} + \Delta a_{22}$, determine the limits for discrete change Δa_{22} so as to maintain the optimum solution.
3. A stainless steel utensil manufacturer makes three types of utensils. The restrictions, profits and requirements are tabulated below :

Utensil Type	*I*	*II*	*III*
Raw material requirement (kg. per unit)	6	3	5
Welding and finishing time (hours per unit)	3	4	5
Profit per unit (Rs.)	3	1	4

If stainless steel (raw material) availability is 25 kg and welding and finishing time available is 20 hours per day, the optimum product mix problem boils down to—

Max. $z = 3x_1 + x_2 + 4x_3$, subject to the constraints :
$6x_1 + 3x_2 + 5x_3 \leq 25$ kg. (raw material restriction)
$3x_1 + 4x_2 + 5x_3 \leq 20$ hours (time restriction)
and x_i, the number of units of ith type to be produced ≥ 0.

Proceeding in the usual way, we get the optimal table as given below.Circle the appropriate answer of the following post-optimality conditions in question regardless of the earlier conditions :

		$c_j \rightarrow$	3	1	4	0	0	
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	X_5	
x_1	3	5/3	1	– 1/3	0	1/3	– 1/3	
x_2	4	3	0	1	1	– 1/5	2/5	
		$z = 17$	0	2	0	1/5	3/5	$\leftarrow \Delta_j$

(a) The second type of utensil would change the currently obtained optimal basis if its profit per unit is :
≥ 1, $\geq 1\frac{1}{5}$, $\geq 1\frac{3}{5}$, ≥ 2, ≥ 3.

(b) The simplex multiplier assiciated with the machine time restriction of 20 hours is $(-3/5)$. Thus multiplier remains unchanged for the upper limit on the machine time availability of

25 $\quad 27\frac{1}{2} \quad$ 35 $\quad 42\frac{1}{2} \quad \geq 42\frac{1}{2}$

(c) The increase in the objective function for each unit availability on machine time higher than upper limit indicated in (b) above is .... (show calculations).

(d) The profit of third type of utensil is Rs. 4.00 per unit. The lower limit on its profitability such that the current basis is still optimal is,

4 $\quad$ 3 $\quad 2\frac{1}{2} \quad$ 2 $\quad < 2$

(e) Suppose the objective function to be maximized is changed to $z = 2x_1 + 3x_2 + 4x_3$. Write the optimal solution, if possible.

(f) If the machine hour availability reduces from 20 hours to 18 hours, the optimal programme would require (i)increase, (ii) decrease in the activity level of type third of utensil, by

2/5 $\quad$ 4/5 $\quad$ 5/4 $\quad$ 2 $\quad$ 5/2 .

4. Solve the LLP : max. $z = 8x_1 + 9x_2$, subject to the constraints :

$$5x_1 + 4x_2 \leq 40, \ x_1 + 2x_2 \leq 12, \ 5x_1 + 19x_2 \leq 95, \text{ and } x_1, x_2 \geq 0.$$

Use sensitivity analysis to modify the optimal solutions as follows :

(i) first add the constraint $4x_1 + 5x_2 \leq 40$. (ii) then delete the constraint $5x_1 + 4x_2 \leq 40$.

What this amounts to ? **[Roorkee M.Sc. I (Appl. Maths.) 96]**

OBJECTIVE QUESTIONS

1. Sensitivity analysis
(a) is also called post-optimality analysis as it is carried out after the optimal solution is obtained.
(b) allows the decision-maker more meaningful information about changes in the LP model parameters.
(c) provides the range within which a parameter may change without affecting optimality.
(d) all of the above.

2. When an additional variable is added in the LP model, the existing optimal solution can further be improved if
(a) $z_j - c_j \leq 0$. (b) $z_j - c_j \geq 0$. (c) both (a) and (b). (d) none of the above.

3. Addition of an additional constraint in the existing constraints will cause a
(a) change in objective function coefficients (c_j). (b) change in coefficients a_{ij}.
(c) both (a) and (b). (d) one of the above.

4. If the additional constraint added is an equation and an artificial variable appears in the basis of the new problem, the new optimal solution is obtained by
(a) assigning zero cost coefficient to the artificial variable if it appears in the basis at negative value
(b) assigning $-M$ cost coefficient to the artificial variable if it appears in the basis at positive value
(c) either (a) or (b). (d) none of the above.

5. To ensure best marginal increase in the objective function value, a resource value may be increased whose shadow price is comparatively
(a) larger. (b) smaller. (c) neither (a) nor (b). (d) both (a) and (b).

6. A non-basic variable should be brought into the new solution mix provided its controbution rate (c_j) is
(a) $c_j^* = c_j + (z_j - c_j)$. (b) $c_j^* > c_j + (z_j - c_j)$. (c) $c_j^* < c_j + (z_j - c_j)$. (d) none of the above.

7. While performing sensitivity analysis, the upper bound infinity on the value of the right hand side of a constraint means that
(a) there is slack in the constraint. (b) the constraint is redundant.
(c) the shadow price for that constraint is zero. (d) none of the above.

8. The entering variable in the sensitivity analysis of objective function coefficients is always a
(a) decision variable. (b) non-basic variable. (c) basic variable. (d) slack variable.

9. To maintain optimality of current optimal solution for a change Δc_k in the coefficient c_k of non-basic variable x_k, we must have
(a) $\Delta c_k = z_k - c_k$. (b) $\Delta c_k = z_k$. (c) $c_k + \Delta c_k = z_k$. (d) $\Delta c_k \geq z_k$.

10. In sensitivity analysis of the coeficient of the non-basic variable in cost minimization LP problem, the upper sensitivity limit is
(a) original value + lowest positive value of improvement ratio.
(b) original value − lowest absolute value of improvement ratio.
(c) positive infinity. (d) negative infinity.

Answers

1. (d) **2.** (a) **3.** (c) **4.** (c) **5.** (a) **6.** (c) **7.** (d) **8.** (d) **9.** (c) **10.** (c).

✱✱✱

10

INTEGER LINEAR PROGRAMMING

(Cutting Plane Method & Branch-and-Bound Method)

10.1. INTRODUCTION

As the name implied *'Integer Linear Programming Problems'* are the special class of linear programming problems where *all* or *some* of the variables in the optimal solution are restricted to non-negative integer values. Such problems are called as ***'all integer'*** or ***'mixed integer'*** problems depending, respectively, on whether all or some of the variables are restriced to integer values.

One might think it sufficient to obtain an integer solution to this special class of linear programming problem by using regular simplex method and then rounding off the fractional values thus occuring in the optimal solution. But in some cases, the deviation from the "exact" optimal integer values (as a result of rounding) may become large enough to give an infeasible solution. Hence there was a need to develop a systematic procedure in order to identify the optimal integer solution to such problems.

In 1956, ***R. E. Gomory*** suggested first of all the systematic method to obtain an optimum integer solution to an *'all integer programming problem'*. Later, he extended the method to deal with the more complicated case of *'mixed integer programming problems'* when only some of the variables are required to be integer. These algorithms are proved to converge to the optimal integer solution in a finite number of iterations making use of familiar dual simplex method. This is called the *"cutting plane algorithm"* because it mainly introduces the clever idea of constructing *"secondary"* constraints which, when added to the optimum (non-integer) solution, will effectively cut the solution space towards the required result. Successive application of these constraints should gradually force the non-integer optimum solution toward the desired *"all-integer"* or *"mixed integer"* solution.

Another important approach, called the ***"branch-and-bound technique"*** for solving both the all-integer and the mixed-integer problems, has originated the straight forward idea of finding all feasible integer solutions. A general algorithm for solving ***'all integer'*** and ***'mixed integer'*** linear programming problems was developed by **A.H. Land** and **A.G. Doig** (1960). Also, **Egon Balas** (1965) introduced an interesting enumerative algorithm for L.P. problem with the variables having the value zero or *one,* called the ***zero one programming problem.***

Several algorithms have been developed so far for solving the integer programming problems. But, in this chapter, we shall discuss only two methods : (i) *Gomory's cutting plane method,* and (ii) *Branch-and-bound method.*

10.2. IMPORTANCE OF INTEGER PROGRAMMING PROBLEMS

We have already pointed out earlier that most industrial applications of large scale programming models are oriented towards planning decisions. There are several frequently occuring circumstances in business and industry that lead to planning models involving integer-valued variables. For example, in production, manufacturing is frequently scheduled in terms of batches, lots or runs. In allocation of goods, a shipment must involve a discrete number of trucks, freight cars or aircrafts. In such cases, the fractional value of variables may be meaningless in the context of the actual decision problem. For example, it is not possible to use 2.5 boilers in a thermal power station, 10.4 men in a project, or 5·6 lathes in a workshop.

Many other decision problems can necessitate integer programming models. One category of such problems deals with the sequencing, scheduling and routing decisions. An example is the ***travelling salesman problem*** which aims at finding a least distance route for a salesman who must visit each of n cities, starting and ending his journey at home city. Larger expenditures of capital and resources are required in *capital budgeting* decisions. This is the main reason why integer programming is so important for marginal decisions. An optimal solution to a capital budgeting problem may yield considerably more profit to a firm than will an approximate or guessed-at solution. For example, fertilizer manufacturing firm with 15 plants may be able to substantially increase profits by cutting back to 10 plants or less, provided this reduction is planned optimally.

A linear programming problem in which some or all variables $x_1 \ldots x_n$ are permitted to take the integral values (whole numbers), is referred as an integer (or discrete) programming problem (I.P.P.). Mathematical model of the integer programming problem is as follows :

$$\left.\begin{array}{lll} \text{Optimize :} & \sum_{j=1}^{n} c_j x_j , & \\ \text{Subject to :} & \sum_{j} a_{ij} x_j = b_i, & i = 1, 2, \ldots, m, \\ \text{and} & x_j \geq 0, j = 1, 2, \ldots, n & \\ \text{and } x_j \text{ interger valued, for} & j = 1, 2, \ldots p \leq n. & \end{array}\right\} \quad \ldots(A)$$

An I.P.P. is termed as pure I.P.P. if the all variables are restricted to take only integral values, *i.e.* $p = n$, otherwse if $p < n$ *i.e.*, if only some (say p) variables are restricted to take only integer values, and the remaining $(n - p)$ variables are free to take any non-negative values, then the problem is called a *mixed I.P.P.*, Since in an I.P.P. variables are restricted to take discrete values of the variables, the function (objective as well as functions involve in the constraints) in an I.P.P. are defined only at discrete values of variables and it is also called as disorded programming problem. The integer programming problem in which variables can take non-negative values continuously, is termed continuous programming problem. If we drop the last restriction requiring x_j integer-valued, the problem becomes a continuous programming problem. Further 'x_j *integer value*' is only the restriction which distinguishes L.P.P. and integer programming problem.

Constraints in an I.P.P. may include any of the sign, (≤) or (≥) or (=). But by introducing slack and / or surplus variables, we can always convert them into strict equations. That is why in (A) constraints are $\Sigma a_{ij} x_j = b_i$. Actually the integer programming algorithm does not differentiate between the original and slack variables in the sense that all variables must be integers. One such problem is capital budgeting problem. Consider the problem of a firm which has n projects to undertake. But due to budget limitations, not all can be undertaken. Let the present value of jth project be c_j ($j = 1, ,2 \ldots, n$). If b_i ($i = 1, 2, \ldots\ldots, m$) be the amount of capital available in period l, then the firm's problem mathematically becomes as follows :

$$\begin{array}{lll} \text{Maximize :} & z = \sum_{j=1}^{n} c_j x_j & \\ \text{Subject to :} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i , \; i = 1, 2, \ldots, m & \\ \text{and} & x_j = 0 \text{ or } 1, j = 1, 2, \ldots, n & \\ \text{where} & x_j = 1, & \text{if project } j \text{ is selected} \\ & = 0, & \text{if project } j \text{ is not selected} \end{array}$$

In business situations it often arise where the variables of interest have to be integers. Consider for example, the product-mix problem, where a company, operating within the existing departmental capacities, has to decide on the number of units of each product to be manufactured, so as to maximize the profit. Under certain assumptions, it is possible to formulate the situations as a linear program and to solve it by using any linear programming (L.P.) technique. The optimal solution in such cases may result in fractional values of the decision variables. But such fractional values do not make any sense in practice, and as such, one is tempted to round-off these values to the nearest integers and use them for action. Rounding off, may thus result is sub-optimal or infeasible solution.

It is possible to take care of such eventualities in the initial values. Such a *linear program with decision variables restricted to integer values is called integer program.*

While the integer restrictions on decision variables in a L.P. framework may be inherent in the problem situation, there are also situations, where these restrictions may be imposed by the analyst. Integer variables, specifically those which can take a value of zero or one, have been used in many problems to provide modelling capabilities beyond those available in L.P. Consider for example, the capital budgeting problem, where a company is to choose m among n projects. As each project has an associated return and uses a specific quantity of one or more resources, the problem is to choose the projects so as to maximize return within the given resource constraint. Analysts have used integer variables to formulate this problem. The decision variables are defined as x_j corresponding to each project j and x is allowed to take a value of one or zero depending on n whether the project j is chosen or not. Once again, if the integer restrictions are relaxed and the problem is solved by L.P. technique, fractional values of x_j' s may result, which will be meaningless in this case.

The possible consequences if one solves an integer linear programming problem ignorning the 'integer constraints' and then rounds of the non-integer values of the optimum solution to integers are as given below :

One may be tempted to solve an integer linear programming problem ignoring the 'interger constraints' and then round off the non-integer values of the optimum solution to integers. But there is no guarrantee that the integer-valued solutions that obtained will satisfy all the constraints and as such the solution may not be feasible at all.

Also the solution with rounded off integer values may not now give the optimum solution. At most it may give a sub-optimal solution. Due to these reasons, special techniques have been designed for obtaining an optimum solution in integer programming itself.

In many industrial and business problems, variables have to be restricted to integers. Such situations are :

(i) Machine scheduling problems where machines can obviously take only integer values.

(ii) The manufacture of refrigerators, cars, T.V. sets etc. involves integer values only, as a fraction of TV set or car cannot be produced.

(iii) Assignment of duties to persons in a department is another example of integer restriction.

In general, in any situation involving the decision of the type "either-or" can be viewed as an integer programming problem.

Q. What is an integer linear programming problem ? How does the optimal solution of integer programming problem compare with that of linear programming problem? **[IGNOU 2000 (June)]**

10.3. WHY INTEGER PROGRAMMING IS NEEDED ?

We might think it sufficient to obtain an integer solution to the given LP problem by first obtaining the non-integer optimal solution using regular simplex method (or graphical method) and then rounding off the fractional values of the variables. But, in some cases, the deviation from the "exact" optimal integer solution (obtained as a result of rounding) may become large enough to give an infeasible solution. Hence it was necessary to develop a systematic procedure to determine the *optimal integer* solution to such problems. The following example will make the concept more clear.

The question *"why integer programming is needed ?"* can be more easily answered through the following illustrative example.

Consider a simple problem : Max. $z = 10x_1 + 4x_2$ *, subject to the constraints :*

$$3x_1 + 4x_2 \leq 8\ ,\ x_1 \geq 0,\ x_2 \geq 0,\ \text{and } x_1\ ,\ x_2 \text{ are integers.}$$

First, ignoring the integer valued restrictions, we obtain the optimal solution :

$x_1 = 2\frac{2}{3}$, $x_2 = 0$, max. $z = 26\frac{2}{3}$, by using graphical method. Then, by rounding off the fractional value of $x_1 = 2\frac{2}{3}$, the optimum solution becomes $x_1 = 3$, $x_2 = 0$ with max. $z = 30$. But this solution does not satisfy the constraint $3x_1 + 4x_2 \leq 8$ and thus this solution is not feasible.

Now, again, if we round off the solution to $x_1 = 2$, $x_2 = 0$, obviously this is the feasible solution and also integer valued. But, this solution gives $z = 20$ which is far away from the optimum value of $z = 26\frac{2}{3}$. So, this is another disadvantage of getting an integer valued solution by *rounding down* the exact optimum solution. Still there is no guarantee that the

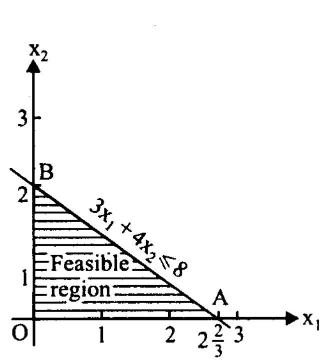


Fig. 10.1

"rounding down" solution will be an optimal one because it may be far away from the optimum solution.

Thus, a systematic procedure for obtaining an exact optimum integer solution to integer programming problems is needed.

We shall now give the formal definitions of integer programming problems.

10.4. DEFINITIONS

1. ***Integer Programming Problem (I.P.P.).*** *The linear programming problem : Max* $z = \mathbf{CX}$ *, subject to* $\mathbf{AX} = \mathbf{b}, \mathbf{X} \geq \mathbf{0}$ *and some* $x_j \in \mathbf{X}$ *are integers, where* $\mathbf{C}, \mathbf{X} \in R^n$ *,* $\mathbf{b} \in R^m$ *and* $\mathbf{A}$ *is an* $m \times n$ *real matrix, is called an Integer Programming Problem, abbreviated as I.P.P.*

2. ***All Integer Programming Problem (All I.P.P.).*** An integer programming problem is said to be an *"All Integer Programming Problem"* if all $x_j \in \mathbf{X}$ are integers.

3. ***Mixed Integer Programming Problem (Mixed I.P.P.).*** An integer programming problem is said to be *'Mixed Integer Programming Problem'* if not all $x_j \in \mathbf{X}$ are integers.

Q. 1. State the general form of an integer programming problem. [Meerut M.Sc. (Math) 93]
2. Distinguish between pure and mixed integer programming problems.

I–Gomory's Cutting Plane Method

10.5. GOMORY'S ALL INTEGER PROGRAMMING TECHNIQUE

In this technique, we first find the optimum solution of the given I.P.P. by regular simplex method as discussed earlier, disregarding the integer condition of variables. Then, we observe the following :

(i) If all the variables in the optimum solution thus obtained have integer values, the current solution will be the desired optimum integer solution.

(ii) If not, the considered L.P.P. requires modification by introducing a *secondary* constraint (also called *Gomory's Constraint*) that reduces some of the non-integer values of variables to integer one, but does not eliminate any feasible integer.

(iii) Then, an optimum solution to this modified L.P.P. is obtained by using any standard algorithm. If all the variables in this solution are integers, then the optimum integer solution is obtained. Otherwise, another *secondary constraint* is added to the L.P.P. and the entire procedure is repeated.

In this way, the optimum integer solution will be obtained eventually after introducing the sufficient number of new constraints. Thus, it becomes specially important to discuss below— how the additional constraints (*Gomory's Constraints*) are constructed.

10·5–1. How to Construct Gomory's Constraint

The secondary constraints which will force the solution toward an all-integer point are constructed as follows.

Let the optimum non-integer solution to the maximization L.P.P. has been obtained. In our usual notations, this solution can be shown by the following optimal simplex table.

Table 10·1

			BASIC						NON BASIC			
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X}_1(\beta_1)$	$\mathbf{X}_2(\beta_2)$	...	$\mathbf{X}_i(\beta_i)$	...	$\mathbf{X}_m(\beta_m)$	$\mathbf{X}_{m+1}$	...	$\mathbf{X}_n$	
x_1	c_{B1}	x_{B1}	1	0	...	0	...	0	$x_{1,m+1}$	...	x_{1n}	
x_2	c_{B2}	x_{B2}	0	1	...	0	...	0	$x_{2,m+1}$	...	x_{2n}	
:	:	:	:	:		:		:	:		:	
x_i	c_{Bi}	x_{Bi}	0	0	...	1	...	0	$x_{i,m+1}$	...	x_{in}	
:	:	:	:	:	:	:	:	:	:	:	:	
x_m	c_{Bm}	x_{Bm}	0	0	...	0	...	1	$x_{m,m+1}$	..	x_{mn}	
	$z = \mathbf{C_B X_B}$		0	0	...	0	...	0	Δ_{m+1}	...	Δ_n	$\leftarrow \Delta_j$

In this table, the variable (x_{Bi}, $i = 1, 2, ..., m$) represent the basic variables and the remaining $(n - m)$ variables $x_{m+1}, x_{m+2}, \ldots, x_n$ are the non-basic variables. However, these variables have been arranged in this order, for our convenience.

Let the ith basic variable x_{Bi} possesses a non-integer value which is given by the constraint equation

$$x_{Bi} = 0x_1 + 0x_2 + \ldots + 1x_i + \ldots + 0x_m + x_{i,m+1}\, x_{m+1} + \ldots + x_{in}\, x_n$$

or $$x_{Bi} = x_i + \sum_{j=m+1}^{n} x_{ij}\, x_j \qquad \text{or} \qquad x_i = x_{Bi} - \sum_{j=m+1}^{n} x_{ij}\, x_j \qquad ...(10\cdot1)$$

Now, let $\boxed{x_{Bi} = I_{Bi} + f_{Bi}}$ and $x_{ij} = I_{ij} + f_{ij}$, where I_{Bi} and I_{ij} are the largest integral parts of x_{Bi} and x_{ij}, respectively, such that $I_{Bi} \le x_{Bi}$ and $I_{ij} \le x_{ij}$. It follows that $0 < f_{Bi} < 1$ and $0 \le f_{ij} < 1$; that is, f_{Bi} is strictly positive fraction while f_{ij} is a non-negative fraction. For example,

x_a	I_a	$f_a = x_a - I_a$
$2\frac{1}{2}$	2	$\frac{1}{2}$
$-1\frac{1}{3}$	-2	$\frac{2}{3}$
-2	-2	0
$-\frac{3}{5}$	-1	$\frac{2}{5}$

Now substituting above values in the eqn. (10·1) for x_i, we get

$$x_i = (I_{Bi} + f_{Bi}) - \sum_{j=m+1}^{n} (I_{ij} + f_{ij})\, x_j \qquad ...(10\cdot2)$$

or $$f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j = x_i - I_{Bi} + \sum_{j=m+1}^{n} I_{ij}\, x_j\,. \qquad ...(10\cdot3)$$

Now for all the variables x_i ($i = 1, 2, ..., m$) and x_j ($j = m + 1, ..., n$) to be integer valued, the right hand side of the above equation must be an integer. This implies that left-hand side

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j$$

must also be an integer. Since $0 < f_{Bi} < 1$ and $\sum_{j=m+1}^{n} f_{ij}\, x_j \ge 0$, it follows that the inequality condition is satisfied if

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j \le 0\,. \qquad ...(10\cdot4)$$

This is true because $f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j \le f_{Bi} < 1$.

But, since $f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j$ is an integer, then it can be either a zero or a negative integer.

Now the constraint (10·4) can be put in the form

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j + g_i = 0\,, \qquad \text{or} \qquad \boxed{-f_{Bi} = -\sum_{j=m+1}^{n} f_{ij}\, x_j + g_i} \qquad ...(10\cdot5)$$

where g_i is a non-negative *Gomorian slack variable* which by definition must also be an integer. The constraint equation (10·5) defines the so-called *Gomory's cutting plane.* From *Table* 10·1, the non-basic variables $x_j = 0$ ($j = m + 1, \ldots, n$) and thus by virtue of (8·5) $g_i = -f_{Bi}$ which is clearly infeasible. Thus in order to clear this infeasibility, we have no alternative except to use the ***dual simplex method*** (as described in *chapter 6*). Practically, this is equivalent to cutting of the solution space towards the optimal integer solution.

Now, after adding the *Gomory's Constraint* (10·5), the optimum simplex Table 10·1 takes the form :

Table 10·2. Addition of Gomory's Constraint

BASIC VAR.	X_B	X_1 (β_1)	X_2 (β_2)	...	X_i (β_i)	...	X_m (β_m)	X_{m+1}	...	X_n	G_1 (β_{m+1})	
x_1	x_{B1}	1	0	...	0	...	0	$x_{1,m+1}$	...	x_{1n}	0	
x_2	x_{B2}	0	1	...	0	...	0	$x_{2,m+1}$	...	x_{2n}	0	
:	:	:	.	:	:	:	:	:	:	:	:	
x_i	x_{Bi}	0	0	..	1	...	0	$x_{i,m+1}$	...	x_{in}	0	
:	:	:	:					:		:	:	
x_m	x_{Bm}	0	0	...	0	...	1	$x_{m,m+1}$	...	x_{mn}	0	
g_i	$-f_{Bi}$	0	0	...	0	...	1	$-f_{B,m+1}$	...	$-f_{Bm}$	1	
	z	0	0	...	0	...	0	Δ_{m+1}	...	Δ_n	...0	$\leftarrow \Delta_j$

If the new solution (after applying the *dual simplex method*) is all-integer one, the process ends. Otherwise, *second Gomory's Constraint* is constructed from the resulting optimal table and the dual simplex method is again used to clear the infeasibility. This process is repeated so long as an all integer solution is obtained. However, if at any iteration, the dual simplex algorithm indicates that no feasible solution exists then the problem has no feasible integer solution.

10·5-2. Gomory's Cutting-plane (All I.P.P.) Algorithm.

The step-by-step procedure for the solution of all-integer programming problem is as follows :

Step 1. If the I.P.P. is in the minimization form, convert it to maximization form.

Step 2. Then convert the inequalities into equations by introducing *slack* and/or *surplus* variables (if necessary) and obtain the optimum solution of the L.P.P. (after ignoring the integer condition) by usual *simplex method.*

Step 3. Now, test the integerality of the optimum solution which is obtained in *Step* 2.

(i) If the optimum solution contains all integer values, then an optimum integer basic feasible solution has been achieved.

(ii) If not, go to next step.

Step 4. Examine the constraint equations corresponding to the current optimal solution. Let these constraints be expressed by $x_{Bi} = x_i + \sum_{j=m+1}^{n} x_{ij}\, x_j \quad (i = 1, 2, ..., m).$

Select the largest fraction of x_{Bi}'s, *i.e.* find $\max_i [f_{Bi}]$. Let it be f_{Bk} for $i = k$.

Step 5. Express the negative fraction, if any in the kth row of the optimum simplex table, as the sum of a negative integer and a non-negative fraction.

Step 6. At this stage, construct the *Gomorian Constraint :*

$$f_{Bi} - \sum_{j=m+1}^{n} f_{ij}\, x_j \le 0,$$

as described in the preceding section, and then introduce the *Gomorian* equation

$$-f_{Bi} = -\sum_{j=m+1}^{n} f_{ij}\, x_j + g_i$$

to the current set of equality constraints.

Step 7. Starting with this new set of constraint equations, obtain the new optimum solution by using *dual simplex method* in order to clear infeasibility. The slack variable g_i will be the initial leaving basic variable.

Step 8. Now two possibilities may arise :

(i) If this new optimum solution for the *Modified* L.P.P. is an all-integer solution, it is also feasible and optimum for the given L.P.P.

(ii) Otherwise, we return to *Step 4* and repeat the entire process until an optimum feasible integer solution is obtained.

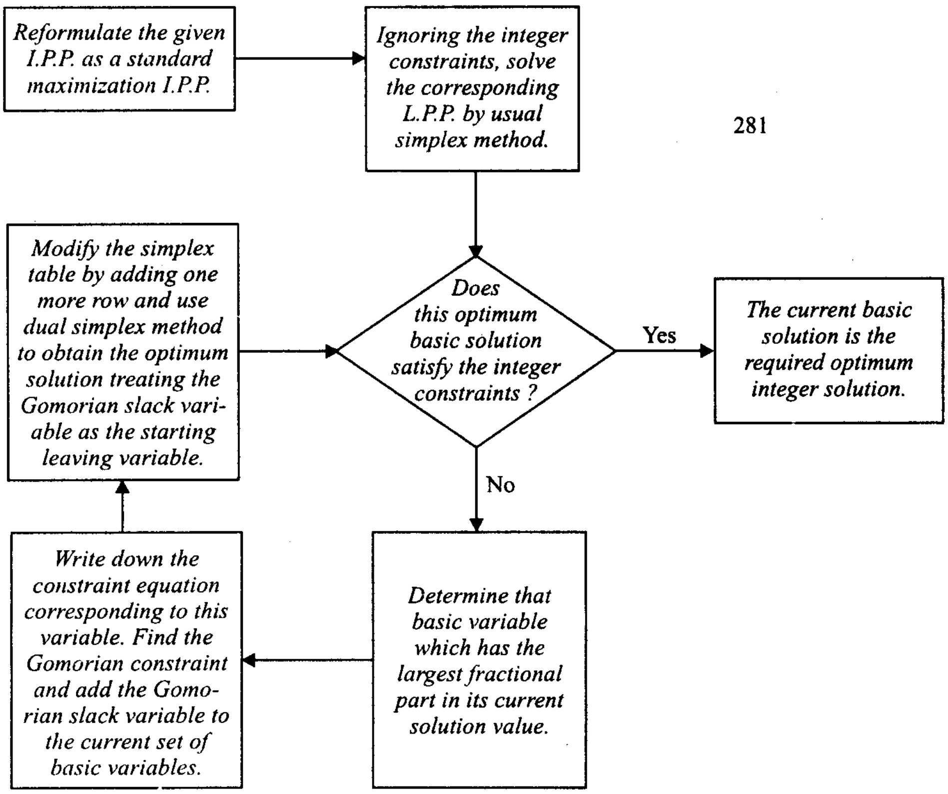
All above steps of Gomory's algorithm can be more precisely demonstrated by the following flow chart :

Q. 1. Explain the concept of integer programming by a suitable example. Give any approach to solve an integer programming problem. **[Madurai B.Sc. (Comp. Sc.) 92]**

2. Explain the algorithm involved in the iterative solution to all I.P.P.

3. Explain the cutting plane method of solving an integer problem.

FLOW CHART OF GOMORY'S ALL I.P.P. ALGORITHM



10·5-3 Computational Demonstration of Gomory's Algorithm

Example 1. *Solve the integer programming problem :*

Max. $z = 7x_1 + 9x_2$, *subject to* $-x_1 + 3x_2 \leq 6,\ 7x_1 + x_2 \leq 35,\ x_1 \geq 0, x_2 \geq 0,$ *and integers.*

[Kanpur 96]

Solution. Step 1. Since the problem is already given in standard maximization form, we go to the next step.

Step 2. Introducing the slack variables, we get the constraint equations

$$\begin{aligned} -x_1 + 3x_2 + x_3 \qquad &= 6 \\ 7x_1 \ \ + x_2 \qquad + x_4 &= 35 . \end{aligned}$$

Now ignoring the integer conditions and then using the regular simplex method we get the following set of tables. The optimum solution to the L.P.P. is given by *Table* 10·3 .

Table 10·3

BASIC VARIABLES	C_B	X_B	X_1 (7)	X_2 (9)	X_3 (0)	X_4 (0)	MIN. (X_B/X_k)
$\leftarrow x_3$	0	6	-1	$\boxed{3}$ ↑	1	0	$6/3 \leftarrow$
x_4	0	35	7	1	0	1	35/1
	$z = C_B X_B = 0$		-7	-9 ↑	0 ↓	0	$\leftarrow \Delta_j$
x_2	9	2	$-1/3$	1	1/3	0	—
$\leftarrow x_4$	0	33	$\boxed{22/3}$	0	$-1/3$	1	$33/\frac{22}{3} \leftarrow$
	$z = 18$		-10 ↑	0	3	0 ↓	$\leftarrow \Delta_j$
x_2	9	$3\frac{1}{2}$	0	1	7/22	1/22	
x_1	7	$4\frac{1}{2}$	1	0	$-1/22$	3/22	
	$z = C_B X_B = 63$		0	0	28/11	15/11	$\leftarrow \Delta_j$

The optimum solution thus obtained is : $x_1 = 4\frac{1}{2}$, $x_2 = 3\frac{1}{2}$, $z = 63$.

Step 3. Since the optimum solution obtained as above is not an integer solution because of $x_1 = 4\frac{1}{2}$ and $x_2 = 3\frac{1}{2}$, we go to next step.

Step 4. We now select the constraint corresponding to max (f_{Bi}) = max. (f_{B1}, f_{B2}).

Since $x_{B1} = I_{B1} + f_{B1} = 3 + \frac{1}{2}$, and $x_{B2} = I_{B2} + f_{B2} = 4 + \frac{1}{2}$, we have $f_{B1} = f_{B2} = \frac{1}{2}$.

Hence, max. (f_{B1}, f_{B2}) = max. $\left[\frac{1}{2}, \frac{1}{2}\right] = \frac{1}{2}$.

Thus, in this problem, since both the equations have the same value of f_{Bi}, that is, $f_{B1} = f_{B2} = \frac{1}{2}$, either one of the two equations can be used. Let us consider the x_2-equation, *i.e.*, first-row of optimum table.

Step 5. Negative fraction does not exist.

Step 6. ***To Construct the Gomorian Constraint.***

The Gomorian constraint is given by, $-f_{Bi} = -\sum_{j=m+1}^{n} f_{ij} x_j + g_i$

Here $m = 2$, $n = 4$, $i = 1$, $f_{B1} = \frac{1}{2}$. Thus above constraint becomes :

$-f_{B1} = -\sum_{j=3}^{4} f_{ij} x_j + g_1$ or $-f_{B1} = -f_{13}x_3 - f_{14}x_4 + g_1$ (since x_3, x_4 are slack variables)

Substituting the values : $f_{13} = 7/22$, $f_{14} = 1/22$, $f_{B1} = 1/2$, we get the required *Gomorian Constraint* as

$$-\frac{1}{2} = -\frac{7}{22}x_3 - \frac{1}{22}x_4 + g_1 \qquad (x_3 = x_4 = 0\text{, being non-basic})$$

Obviously, the coefficients of remaining variables x_1 and x_2 in the above *Gomorian constraint* will be taken 0. Thus complete *Gomorian Constraint* can be written as

$$-\frac{1}{2} = 0x_1 + 0x_2 - \frac{7}{22}x_3 - \frac{1}{22}x_4 + g_1$$

Adding this new constraint to the *Optimal Table* 10·3, we get the new *Table* 10·4.

Table 10·4

BASIC VARIABLES	C_B	X_B	X_1 (7)	X_2 (9)	X_3 (0)	X_4 (0)	G_1 (0)	
x_2	9	$3\frac{1}{2}$	0	1	7/22	1/22	0	
x_1	7	$4\frac{1}{2}$	1	0	$-1/22$	3/22	0	
g_1	0	$\rightarrow -1/2$	0	0	$\boxed{-7/22}$	$-1/22$	1	
	$z = C_B X_B = 63$		0	0	28/11 ↑	15/11	0 ↓	$\leftarrow \Delta_j$

Step 7. ***To apply dual simplex method.***

(i) leaving vector is G_1, *i.e.*, β_3. Therefore $r = 3$.

(ii) Entering vector is obtained by

$$\frac{\Delta_k}{x_{rk}} = \max.\left[\frac{\Delta_3}{x_{33}}, \frac{\Delta_4}{x_{34}}\right] = \max\left[\frac{28/11}{-7/22}, \frac{15/11}{-1/22}\right] = \max.\ [-8, -30] = -8 = \frac{\Delta_3}{x_{33}}.$$

Therefore, $k = 3$. Hence we must enter the vector $\mathbf{a}_3$ corresponding to which $\mathbf{X}_3$ is given in the above table. Thus, we get the following transformed table.

Table 10·5

		$c_j \rightarrow$	7	9	0	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{G_1}$	
x_2	9	3	0	1	0	0	1	
x_1	7	$4\frac{4}{7}$	1	0	0	1/7	−1/7	
x_3	0	$1\frac{4}{7}$	0	0	1	1/7	−22/7	
	$z = \mathbf{C_B X_B} = 59$		0	0	0	1	8	$\leftarrow \Delta_j$

$$\Delta_4 = \mathbf{C_B X_4} - c_4 = (9, 7, 0)\,(0, \tfrac{1}{7}, \tfrac{1}{7}) - 0 = (0 + 1 + 0) = 1$$

$$\Delta_5 = \mathbf{C_B G_1} - c_5 = (9, 7, 0)\,(1, -\tfrac{1}{7}, -\tfrac{22}{7}) - 0 = (9 - 1 + 0) = 8.$$

The non-integer optimum solution given by above table is : $x_1 = 4\frac{4}{7}$, $x_2 = 3$, $x_3 = 1\frac{4}{7}$, $z = 59$.

Step 8. The optimal solution as obtained above by dual simplex method is still non-integer. Thus, a new Gomory's constraint is to be constructed again. Selecting x_1-equation (*i.e.*, IInd row of above table) to generate the cutting plane (because largest fractional part can be $f_{B2} = f_{B3} = \frac{4}{7}$), we get the Gomory's constraint as

$$-\tfrac{4}{7} = -\tfrac{1}{7}x_4 - \tfrac{6}{7}g_1 + g_2. \qquad \text{or} \qquad -\tfrac{4}{7} = 0x_1 + 0x_2 + 0x_3 - \tfrac{1}{7}x_4 - \tfrac{6}{7}g_1 + g_2$$

Adding this constraint to the above Table 10·5, we get Table 10·6.

Table 10·6

		$c_j \rightarrow$	7	9	0	0	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{G_1}$	$\mathbf{G_2}$	
x_2	9	3	0	1	0	0	1	0	
x_1	7	$4\frac{4}{7}$	1	0	0	1/7	−1/7	0	
x_3	0	$1\frac{4}{7}$	0	0	1	1/7	−22/7	0	
g_2	0	−4/7	0	0	0	$\boxed{-1/7}$	−6/7	1	
	$z = \mathbf{C_B X_B} = 59$		0	0	0	1 ↑	8	0 ↓	$\leftarrow \Delta_j$

We again apply *dual simplex method.*

(i) Leaving vector is G_2 (*i.e.* β_4). Therefore, $r = 4$.

(ii) Entering vector will be obtained by

$$\frac{\Delta_k}{x_{4k}} = \max.\left[\frac{\Delta_4}{x_{44}}, \frac{\Delta_5}{x_{45}}\right] = \max.\left[\frac{1}{-1/7}, \frac{8}{-6/7}\right] = \max.\ [-7, -9\tfrac{1}{3}] = -7 = \frac{\Delta_4}{x_{44}}.$$

Therefore, $k = 4$. Hence we must enter $\mathbf{a}_4$ corresponding to which $\mathbf{X}_4$ given in the above table. Thus we get the transformed table as below :

Table 10·7

		$c_j \rightarrow$	7	9	0	0	0	0	
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	G_1	G_2	
x_2	9	3	0	1	0	0	1	0	
x_1	7	4	1	0	0	0	−1	1	
x_3	0	1	0	0	1	0	−4	1	
x_4	0	4	0	0	0	1	6	−7	
	$z = C_B X_B = 55$		0	0	0	0	2	7	$\leftarrow \Delta_j$

$$\Delta_5 = C_B G_1 - c_5 = (9, 7, 0, 0)\,(1, -1, -4, 6) - 0 = (9 - 7 + 0 + 0) = 2$$
$$\Delta_6 = C_B G_2 - c_6 = (9, 7, 0, 0)\,(0, 1, 1, -7) - 0 = (0 + 7 + 0 + 0) = 7\,.$$

Thus, finally we get the optimal integer solution : $x_1 = 4$, $x_2 = 3$, max $z = 55$.

Verification by graphical method :

It can be easily verified that the addition of the above Gomory's constraints effectively 'cut' the solution space as desired. Thus the Gomory's first constraint :

$$-\frac{7}{22}x_3 - \frac{1}{22}x_4 + g_1 = -\frac{1}{2},$$

can be expressed in terms of x_1 and x_2 only by substituting :

$$x_3 = 6 + x_1 - 3x_2 \quad \text{and} \quad x_4 = 35 - 7x_1 - x_2$$

from the original constraint equations treating g_1 as a slack variable in *step* 2.

This gives $g_1 + x_2 = 3$ or $x_2 \le 3$, treating g_1 as a slack variable.

Similarly, for the Gomory's second constraint, $-\frac{1}{7}x_3 - \frac{6}{7}g_1 + g_2 = -\frac{4}{7}$, the equivalent constraint in terms of x_1 and x_2 is obtained as $x_1 + x_2 \le 7$.

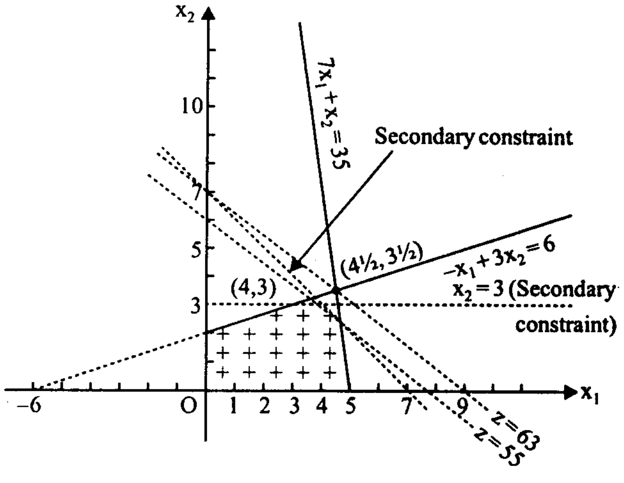


Fig. 10.2

Now plotting the *Gomory's* constraints $x_2 \le 3$ and $x_1 + x_2 \le 7$ in addition to the constraints of the given problem, we find that it will result in the new (optimal) extreme point (4, 3) as shown in Fig. 10·2 .

Example 2. *Find the optimum integer solution to the following all I.P.P :*

Max. $z = x_1 + 2x_2$, *subject to the constraints* $2x_2 \le 7$, $x_1 + x_2 \le 7$, $2x_1 \le 11$, $x_1 \ge 0$, $x_2 \ge 0$, *and* x_1 , x_2 *are integers.* **[Vidyasagar 97]**

Solution. Step 1. Introducing the slack variables, we get

$$\begin{aligned} 2x_2 + x_3 \quad\quad\quad &= 7 \\ x_1 + x_2 \quad + x_4 \quad &= 7 \\ 2x_1 \quad\quad\quad + x_5 &= 11 \\ x_1\,, x_2\,, x_3\,, x_4\,, x_5 &\ge 0\,. \end{aligned}$$

Step 2. Ignoring the integer condition, we get the initial simplex table as follows :

Table 10·8

		$c_j \rightarrow$	1	2	0	0	0	
BASIC VAR.	C_B	X_B	X_1	X_2	X_3	X_4	X_5	MIN RATIO (X_B/X_2)
$\leftarrow x_3$	0	7	0	$\boxed{2}$	1	0	0	7/2 ←
x_4	0	7	1	1	0	1	0	7/1
x_5	0	11	2	0	0	0	1	—
	$z = 0$		−1	−2 ↑	0 ↓	0	0	$\leftarrow \Delta_j$

Introducing X_2 and leaving X_3 from the basis, we get

Table 10·9

BASIC VAR.	C_B	$c_j \rightarrow$ X_B	1 X_1	2 X_2	0 X_3	0 X_4	0 X_5	MIN (X_B/X_k)
x_2	2	$3\frac{1}{2}$	0	1	1/2	0	0	—
$\leftarrow x_4$	0	$3\frac{1}{2}$	$\leftarrow \boxed{1}$	0	$-1/2$	1	0	$3\frac{1}{2}/1 \leftarrow$
x_5	0	11	2	0	0	0	1	11/2
	$z = C_B X_B = 7$		-1 $\uparrow$	0	1	0 $\downarrow$	0	$\leftarrow \Delta_j$

$$\Delta_1 = C_B X_1 - c_1 = (2, 0, 0)\,(0, 1, 2) - 1 = -1,\ \Delta_3 = C_B X_3 - c_3 = (2, 0, 0)\,(\tfrac{1}{2}, -\tfrac{1}{2}, 0) - 0 = 1\,.$$

Introducing X_1 and leaving X_4, we get the following optimum table.

Optimum Table 10·10

BASIC VAR.	C_B	$c_j \rightarrow$ X_B	1 X_1	2 X_2	0 X_3	0 X_4	0 X_5	
x_2	2	$3\frac{1}{2}$	0	1	1/2	0	0	
x_1	1	$3\frac{1}{2}$	1	0	$-1/2$	1	0	
x_5	0	4	0	0	1	-2	1	
	$z = 10\frac{1}{2}$		0	0	1/2	1	0	$\leftarrow \Delta_j$

$$\Delta_3 = C_B X_3 - c_3 = (2, 1, 0)\,(\tfrac{1}{2}, -\tfrac{1}{2}, 1) - 0 = (1 - \tfrac{1}{2} + 0) = \tfrac{1}{2}$$
$$\Delta_4 = C_B X_4 - c_4 = (2, 1, 0)\,(0, 1, -2) - 0 = (0 + 1 + 0) = 1\,.$$

The optimum solution thus obtained is : $x_1 = 3\frac{1}{2}$, $x_2 = 3\frac{1}{2}$, $z = 10\frac{1}{2}$.

Step 3. Since the optimum solution obtained above is not an integer solution, we must go to next step.

Step 4. We now select the constraint corresponding to the criterion

$$\max_i\,(f_{Bi}) = \max\,(f_{B1}, f_{B2}, f_{B3}) = \max\,(\tfrac{1}{2}, \tfrac{1}{2}, 0) = \tfrac{1}{2}\,.$$

Since in this problem, the x_2-equation and x_1-equation both have the same value of f_{Bi}, *i.e.* $\frac{1}{2}$, either one of the two equations can be used. Let us consider the *first-row* of the above optimum table. The *Gomory's* constraint to be added is therefore

$$-\sum_{j=3,4} f_{1j}\,x_j + g_1 = -f_{B1} \qquad \text{or} \qquad -f_{13}x_3 - f_{14}x_4 + g_1 = -f_{B1}$$

$$-\tfrac{1}{2}x_3 - 0x_4 + g_1 = -\tfrac{1}{2} \qquad \text{or} \qquad -\tfrac{1}{2}x_3 + g_1 = -\tfrac{1}{2}\ (x_3 = x_4 = 0)$$

Adding this new constraint to the optimal *Table 10·10*, we get the new *Table 10·11*.

Table 10·11 . New table after adding Gomory constraint

BASIC VAR.	C_B	$c_j \rightarrow$ X_B	1 X_1	2 X_2	0 X_3	0 X_4	0 X_5	0 G_1	
x_2	2	$3\frac{1}{2}$	0	1	1/2	0	0	0	
x_1	1	$3\frac{1}{2}$	1	0	$-1/2$	1	0	0	
x_5	0	4	0	0	1	-2	1	0	
g_1	0	$\rightarrow -1/2$	0	0	$\boxed{-1/2}$	0	0	1	
	$z = C_B X_B = 10\frac{1}{2}$		0	0	1/2 $\uparrow$	1	0	0 $\downarrow$	$\leftarrow \Delta_j$

Step 5. ***To apply dual simplex method.*** Now, in order to remove the infeasibility of the optimum solution : $x_1 = 3\frac{1}{2}$, $x_2 = 3\frac{1}{2}$, $x_5 = 4$, $g_1 = -\frac{1}{2}$, we use the dual simplex method.

(i) Leaving vector is $\mathbf{G}_1$ (*i.e.*, β_4). Therefore, $r = 4$.

(ii) Entering vector is given by

$$\frac{\Delta_k}{x_{4k}} = \text{max.}\left[\frac{\Delta_j}{x_{4j}}, x_{4j} < 0\right] = \text{max.}\left[\frac{\Delta_3}{x_{43}}\right] = \text{max.}\left[\frac{\frac{1}{2}}{-\frac{1}{2}}\right] = \frac{\Delta_3}{x_{43}}.$$

Therefore, $k = 3$. So we must enter $\mathbf{a}_3$ corresponding to which $\mathbf{X}_3$ is given in the above table. Thus, dropping $\mathbf{G}_1$ and introducing $\mathbf{X}_3$ we get the following dual simplex table :

Table 10·12

		$c_j \rightarrow$	1	2	0	0	0	0	
BASIC VAR.	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{G}_1$	
x_2	2	3	0	1	0	0	0	1	
x_1	1	4	1	0	0	1	0	−1	
x_5	0	3	0	0	0	−2	1	2	
x_3	0	1	0	0	1	0	0	−2	
		$z = \mathbf{C_B X_B} = 10$	0	0	0	1	0	1	$\leftarrow \Delta_j$

$\Delta_4 = \mathbf{C_B X_4} - c_4 = (2, 1, 0, 0)\ (0, 1, -2, 0) - 0 = 1$, $\Delta_6 = \mathbf{C_B G_1} - c_6 = (2, 1, 0, 0)\ (1, -1, 2, -2) - 0 = 1$.

This shows that the optimum feasible solution has been obtained in integers. Thus, finally, we get the integer optimum solution to the given I.P.P. as : $x_1 = 4$, $x_2 = 3$, and max $z = 10$.

10·5-4. Short-cut Method for Constructing the Gomory's Constraint

After obtaining the non-integer optimal solution by simplex method, we perform the following step-by-step procedure to construct the Gomory's constraint :

Step 1. In the optimal simplex table (with all $\Delta_j \geq 0$), first select the row corresponding to such basic variable which has the maximum fractional value. If more than one basic variables have the same maximum fractional value, then we can select the row corresponding to either of these basic variables.

In ***Example* 1,** (see ***Table* 10.3** with all $\Delta_j \geq 0$) both the basic variables x_2 and x_1 have the same fractional value (*i.e.* $\frac{1}{2}, \frac{1}{2}$). so we can select either x_2-row or x_1-row. In this case, we have selected x_2-row, *i.e.*

$x_2 \rightarrow 3\frac{1}{2}$	0	1	7/22	1/22

Step 2. In the row selected above we express each number in two parts. First part must be an integer and second part must be a *non-negative* fraction. Thus applying this step to above selected row, we get

$x_2 \rightarrow 3 + (1/2)^*$	$0 + 0^*$	$1 + 0^*$	$0 + (7/22)^*$	$0 + (1/22)^*$

Here non-negative fractional values are marked with '*'

Step 3. Then, we write the *negative* of the fractional values which are marked with '*' in *step* 2. Thus, the new row corresponding to Gomorian slack variable g_1 becomes :

$g_1 \rightarrow -\frac{1}{2}$	0	0	−7/22	−1/22

This row can be directly added to the optimal simplex table with Gomorian slack variable g_1 as additional basic variable and immediately we increase the dimension of basis matrix by introducing one more unit matrix column $\mathbf{G}_1$.

Obviously, above constructed new row will give us the Gormorian constraint :

$$-\tfrac{1}{2} \geq 0x_1 + 0x_2 - \tfrac{7}{22}x_3 - \tfrac{1}{22}x_4 \quad \text{or} \quad 0x_1 + 0x_2 - \tfrac{7}{22}x_3 - \tfrac{1}{22}x_4 + g_1 = -\tfrac{1}{2}.$$

After introducing the new row corresponding to the *Gomorian* constraint, we apply usual dual simplex method to proceed further.

Above outlined prodcedure will be very convenient to apply directly (orally) whenever we need to construct a *Gomory's* constraint

Example 3. *Solve the following integer programming problem :*

Max. $z = 2x_1 + 20x_2 - 10x_3$, *subject to the constraints* :

$2x_1 + 20x_2 + 4x_3 \leq 15$, $6x_1 + 20x_2 + 4x_3 = 20$, *and* $x_1, x_2, x_3 \geq 0$; *and are integers.*

Solve the problem as a (continuous) linear program, then show that it is impossible to obtain feasible integer solution by using simple rounding. Solve the problem using any integer program algorithm.

Solution. Introducing the slack variable $x_4 \geq 0$ and an artificial variable $a_1 \geq 0$, an initial basic feasible solution is $x_4 = 15$ and $a_1 = 20$.

Now computing the net-evaluations (Δ_j) and then using simplex method, the following optimum simplex table is obtained.

Optimal Simplex Table 10·13.

		$c_j \rightarrow$	2	20	−10	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	
x_2	20	5/8	0	1	1/5	3/40	
x_1	2	5/4	1	0	0	−1/4	
	$z = 15$		0	0	14	1	$\leftarrow \Delta_j$

Thus the following non-integer optimum solution is obtained :

$$x_1 = 5/4, x_2 = 5/8, x_3 = 0, \max z = 15.$$

The rounded solution will be $x_1 = 1$, $x_2 = 0$, $x_3 = 0$.

Since this solution satisfies the first constraint only, it is not possible to obtain a feasible solution by using simple rounding. So to obtain the integer-valued solution, we proceed as follows :

$$\text{Max.}\ (f_{B1}, f_{B2}) = \text{Max.}\left(\frac{5}{8}, \frac{1}{4}\right) = \frac{5}{8}.$$

Therefore, from the first row of optimal table, we have

$$\tfrac{5}{8} = 0x_1 + x_2 + \tfrac{1}{5}x_3 + \tfrac{3}{40}x_4$$

or $$(0 + \tfrac{5}{8}) = (0 + 0)\,x_1 + (1 + 0)\,x_2 + (0 + \tfrac{1}{5})\,x_3 + (0 + \tfrac{3}{40})\,x_4$$

The corresponding fractional cut will be $-\tfrac{5}{8} = 0x_1 + 0x_2 - \tfrac{1}{5}x_3 - \tfrac{3}{40}x_4 + g_1$

Now inserting the additional constraint in the optimum simplex table, the following modified table is obtained.

Table 10·14

		$c_j \rightarrow$	2	20	−10	0	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	G_1	
x_2	20	5/8	0	1	1/5	3/40	0	
x_1	2	5/4	1	0	0	−1/4	0	
g_1	0 $\rightarrow$	−5/8	0	0	−1/5	$\boxed{-3/40}$	1	
	$z = 15$		0	0	14	1 $\uparrow$	0 $\downarrow$	$\leftarrow \Delta_j$

First Iteration. Remove G_1 and insert X_4 by *dual simplex method.*